

# The LASSO risk for gaussian matrices

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## Abstract

We consider the problem of learning a coefficient vector  $x_0 \in \mathbb{R}^N$  from noisy linear observation  $y = Ax_0 + w \in \mathbb{R}^n$ . In many contexts (ranging from model selection to image processing) it is desirable to construct a sparse estimator  $\hat{x}$ . In this case, a popular approach consists in solving an  $\ell_1$ -penalized least squares problem known as the LASSO or Basis Pursuit DeNoising (BPDN).

For sequences of matrices  $A$  of increasing dimensions, with independent gaussian entries, we prove that the normalized risk of the LASSO converges to a limit, and we obtain an explicit expression for this limit. Our result is the first rigorous derivation of an explicit formula for the asymptotic mean square error of the LASSO for random instances. The proof technique is based on the analysis of AMP, a recently developed efficient algorithm, that is inspired from graphical models ideas.

## 1 Introduction

Let  $x_0 \in \mathbb{R}^N$  be an unknown vector, and assume that a vector  $y \in \mathbb{R}^n$  of noisy linear measurements of  $x_0$  is available. The problem of reconstructing  $x_0$  from such measurements arises in a number of disciplines, ranging from statistical learning to signal processing. In many contexts the measurements are modeled by

$$y = Ax_0 + w, \quad (1.1)$$

where  $A \in \mathbb{R}^{n \times N}$  is a known measurement matrix, and  $w$  is a noise vector.

The LASSO or Basis Pursuit Denoising (BPDN) is a method for reconstructing the unknown vector  $x_0$  given  $y$ ,  $A$ , and is particularly useful when one seeks sparse solutions. For given  $A$ ,  $y$ , one considers the cost functions  $\mathcal{C}_{A,y} : \mathbb{R}^N \rightarrow \mathbb{R}$  defined by

$$\mathcal{C}_{A,y}(x) = \frac{1}{2} \|y - Ax\|^2 + \lambda \|x\|_1, \quad (1.2)$$

with  $\lambda > 0$ . The original signal is estimated by

$$\hat{x}(\lambda; A, y) = \operatorname{argmin}_x \mathcal{C}_{A,y}(x). \quad (1.3)$$

In what follows we shall often omit the arguments  $A, y$  (and occasionally  $\lambda$ ) from the above notations. We will also use  $\hat{x}(\lambda; N)$  to emphasize the  $N$ -dependence. Further  $\|v\|_p \equiv (\sum_{i=1}^m v_i^p)^{1/p}$  denotes the  $\ell_p$ -norm of a vector  $v \in \mathbb{R}^p$  (the subscript  $p$  will often be omitted if  $p = 2$ ).

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A large and rapidly growing literature is devoted to (i) Developing fast algorithms for solving the optimization problem (1.3); (ii) Characterizing the performances and optimality of the estimator  $\hat{x}$ . We refer to Section 1.3 for an unavoidably incomplete overview.

Despite such substantial effort, and many remarkable achievements, our understanding of (1.3) is not even comparable to the one we have of more classical topics in statistics and estimation theory. For instance, the best bound on the mean square error (MSE) of the estimator (1.3), i.e. on the quantity  $N^{-1}\|\hat{x} - x_0\|^2$ , was proved by Candes, Romberg and Tao [CRT06] (who in fact did not consider the LASSO but a related optimization problem). Their result estimates the mean square error only up to an unknown numerical multiplicative factor. Work by Candes and Tao [CT07] on the analogous *Dantzig selector*, upper bounds the mean square error up to a factor  $C \log N$ , under somewhat different assumptions.

The objective of this paper is to complement this type of ‘rough but robust’ bounds by proving *asymptotically exact* expressions for the mean square error. Our asymptotic result holds almost surely for sequences of random matrices  $A$  with fixed aspect ratio and independent gaussian entries. While this setting is admittedly specific, the careful study of such matrix ensembles has a long tradition both in statistics and communications theory and has spurred many insights [Joh06, Tel99].

Although our rigorous results are asymptotic in the problem dimensions, numerical simulations have shown that they are accurate already on problems with a few hundreds of variables. Further, they seem to enjoy a remarkable *universality* property and to hold for a fairly broad family of matrices [DMM10]. Both these phenomena are analogous to ones in random matrix theory, where delicate asymptotic properties of gaussian ensembles were subsequently proved to hold for much broader classes of random matrices. Also, asymptotic statements in random matrix theory have been replaced over time by concrete probability bounds in finite dimensions. Of course the optimization problem (1.2) is not immediately related to spectral properties of the random matrix  $A$ . As a consequence, universality and non-asymptotic results in random matrix theory cannot be directly exported to the present problem. Nevertheless, we expect such developments to be foreseeable.

Our proofs are based on the analysis of an efficient iterative algorithm first proposed by [DMM09], and called AMP, for approximate message passing. The algorithm is inspired by belief-propagation on graphical models, although the resulting iteration is significantly simpler (and scales linearly in the number of nodes). Extensive simulations [DMM10] showed that, in a number of settings, AMP performances are statistically indistinguishable to the ones of LASSO, while its complexity is essentially as low as the one of the simplest greedy algorithms.

The proof technique just described is new. Earlier literature analyzes the convex optimization problem (1.3) –or similar problems– by a clever construction of an approximate optimum, or of a dual witness. Such constructions are largely explicit. Here instead we prove an asymptotically exact characterization of a rather non-trivial iterative algorithm. The algorithm is then proved to converge to the exact optimum.

## 1.1 Definitions

In order to define the AMP algorithm, we denote by  $\eta : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  the soft thresholding function

$$\eta(x; \theta) = \begin{cases} x - \theta & \text{if } x > \theta, \\ 0 & \text{if } -\theta \leq x \leq \theta, \\ x + \theta & \text{otherwise.} \end{cases} \quad (1.4)$$

The algorithm constructs a sequence of estimates  $x^t \in \mathbb{R}^N$ , and residuals  $z^t \in \mathbb{R}^n$ , according to the iteration

$$\begin{aligned} x^{t+1} &= \eta(A^* z^t + x^t; \theta_t), \\ z^t &= y - Ax^t + \frac{1}{\delta} z^{t-1} \langle \eta'(A^* z^{t-1} + x^{t-1}; \theta_{t-1}) \rangle, \end{aligned} \quad (1.5)$$

initialized with  $x^0 = 0$ . Here  $A^*$  denotes the transpose of matrix  $A$ , and  $\eta'(\cdot; \cdot)$  is the derivative of the soft thresholding function with respect to its first argument. Given a scalar function  $f$  and a vector  $u \in \mathbb{R}^m$ , we let  $f(u)$  denote the vector  $(f(u_1), \dots, f(u_m)) \in \mathbb{R}^m$  obtained by applying  $f$  componentwise. Finally  $\langle u \rangle \equiv m^{-1} \sum_{i=1}^m u_i$  is the average of the vector  $u \in \mathbb{R}^m$ .

As already mentioned, we will consider sequences of instances of increasing sizes, along which the LASSO behavior has a non-trivial limit.

**Definition 1.** *The sequence of instances  $\{x_0(N), w(N), A(N)\}_{N \in \mathbb{N}}$  indexed by  $N$  is said to be a converging sequence if  $x_0(N) \in \mathbb{R}^N$ ,  $w(N) \in \mathbb{R}^n$ ,  $A(N) \in \mathbb{R}^{n \times N}$  with  $n = n(N)$  is such that  $n/N \rightarrow \delta \in (0, \infty)$ , and in addition the following conditions hold:*

- (a) *The empirical distribution of the entries of  $x_0(N)$  converges weakly to a probability measure  $p_{X_0}$  on  $\mathbb{R}$  with bounded second moment. Further  $N^{-1} \sum_{i=1}^N x_{0,i}(N)^2 \rightarrow \mathbb{E}_{p_{X_0}}\{X_0^2\}$ .*
- (b) *The empirical distribution of the entries of  $w(N)$  converges weakly to a probability measure  $p_W$  on  $\mathbb{R}$  with bounded second moment. Further  $n^{-1} \sum_{i=1}^n w_i(N)^2 \rightarrow \mathbb{E}_{p_W}\{W^2\}$ .*
- (c) *If  $\{e_i\}_{1 \leq i \leq N}$ ,  $e_i \in \mathbb{R}^N$  denotes the standard basis, then  $\max_{i \in [N]} \|A(N)e_i\|_2, \min_{i \in [N]} \|A(N)e_i\|_2 \rightarrow 1$ , as  $N \rightarrow \infty$  where  $[N] \equiv \{1, 2, \dots, N\}$ .*

Let us stress that our proof only applies to a subclass of converging sequences (namely for gaussian measurement matrices  $A(N)$ ). The notion of converging sequences is however important since it defines a class of problem instances to which the ideas developed below might be generalizable.

For a converging sequence of instances, and an arbitrary sequence of thresholds  $\{\theta_t\}_{t \geq 0}$  (independent of  $N$ ), the asymptotic behavior of the recursion (1.5) can be characterized as follows.

Define the sequence  $\{\tau_t^2\}_{t \geq 0}$  by setting  $\tau_0^2 = \sigma^2 + \mathbb{E}\{X_0^2\}/\delta$  (for  $X_0 \sim p_{X_0}$  and  $\sigma^2 \equiv \mathbb{E}\{W^2\}$ ,  $W \sim p_W$ ) and letting, for all  $t \geq 0$ :

$$\tau_{t+1}^2 = F(\tau_t^2, \theta_t), \quad (1.6)$$

$$F(\tau^2, \theta) \equiv \sigma^2 + \frac{1}{\delta} \mathbb{E}\{[\eta(X_0 + \tau Z; \theta) - X_0]^2\}, \quad (1.7)$$

where  $Z \sim \mathcal{N}(0, 1)$  is independent of  $X_0$ . Notice that the function  $F$  depends implicitly on the law  $p_{X_0}$ .

We say a function  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is *pseudo-Lipschitz* if there exist a constant  $L > 0$  such that for all  $x, y \in \mathbb{R}^2$ :  $|\psi(x) - \psi(y)| \leq L(1 + \|x\|_2 + \|y\|_2)\|x - y\|_2$ . (This is a special case of the definition used in [BM10] where such a function is called *pseudo-Lipschitz of order 2*.)

Our next proposition that was conjectured in [DMM09] and proved in [BM10]. It shows that the behavior of AMP can be tracked by the above one dimensional recursion. We often refer to this prediction by *state evolution*.

**Theorem 1.1** ([BM10]). *Let  $\{x_0(N), w(N), A(N)\}_{N \in \mathbb{N}}$  be a converging sequence of instances with the entries of  $A(N)$  iid normal with mean 0 and variance  $1/n$  and let  $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a pseudo-Lipschitz function. Then, almost surely*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_i^{t+1}, x_{0,i}) = \mathbb{E} \left\{ \psi(\eta(X_0 + \tau_t Z; \theta_t), X_0) \right\}, \quad (1.8)$$

where  $Z \sim \mathcal{N}(0, 1)$  is independent of  $X_0 \sim p_{X_0}$ .

In order to establish the connection with the LASSO, a specific policy has to be chosen for the thresholds  $\{\theta_t\}_{t \geq 0}$ . Throughout this paper we will take  $\theta_t = \alpha \tau_t$  with  $\alpha$  is fixed. In other words, the sequence  $\{\tau_t\}_{t \geq 0}$  is given by the recursion

$$\tau_{t+1}^2 = F(\tau_t^2, \alpha \tau_t). \quad (1.9)$$

This choice enjoys several convenient properties [DMM09].

## 1.2 Main result

Before stating our results, we have to describe a *calibration* mapping between  $\alpha$  and  $\lambda$  that was introduced in [DMM10].

Let us start by stating some convenient properties of the state evolution recursion.

**Proposition 1.2** ([DMM09]). *Let  $\alpha_{\min} = \alpha_{\min}(\delta)$  be the unique non-negative solution of the equation*

$$(1 + \alpha^2)\Phi(-\alpha) - \alpha\phi(\alpha) = \frac{\delta}{2}, \quad (1.10)$$

with  $\phi(z) \equiv e^{-z^2/2}/\sqrt{2\pi}$  the standard gaussian density and  $\Phi(z) \equiv \int_{-\infty}^z \phi(x) dx$ .

For any  $\sigma^2 > 0$ ,  $\alpha > \alpha_{\min}(\delta)$ , the fixed point equation  $\tau^2 = F(\tau^2, \alpha \tau)$  admits a unique solution. Denoting by  $\tau_* = \tau_*(\alpha)$  this solution, we have  $\lim_{t \rightarrow \infty} \tau_t = \tau_*(\alpha)$ . Further the convergence takes place for any initial condition and is monotone. Finally  $|\frac{dF}{d\tau^2}(\tau^2, \alpha \tau)| < 1$  at  $\tau = \tau_*$ .

For greater convenience of the reader, a proof of this statement is provided in Appendix A.1.

We then define the function  $\alpha \mapsto \lambda(\alpha)$  on  $(\alpha_{\min}(\delta), \infty)$ , by

$$\lambda(\alpha) \equiv \alpha \tau_* \left[ 1 - \frac{1}{\delta} \mathbb{E} \{ \eta'(X_0 + \tau_* Z; \alpha \tau_*) \} \right]. \quad (1.11)$$

This function defines a correspondence (calibration) between the sequence of thresholds  $\{\theta_t\}_{t \geq 0}$  and the regularization parameter  $\lambda$ . It should be intuitively clear that larger  $\lambda$  corresponds to larger thresholds and hence larger  $\alpha$  since both cases yield smaller estimates of  $x_0$ .

In the following we will need to invert this function. We thus define  $\alpha : (0, \infty) \rightarrow (\alpha_{\min}, \infty)$  in such a way that

$$\alpha(\lambda) \in \{ a \in (\alpha_{\min}, \infty) : \lambda(a) = \lambda \}.$$

The next result implies that the set on the right-hand side is non-empty and therefore the function  $\lambda \mapsto \alpha(\lambda)$  is well defined.

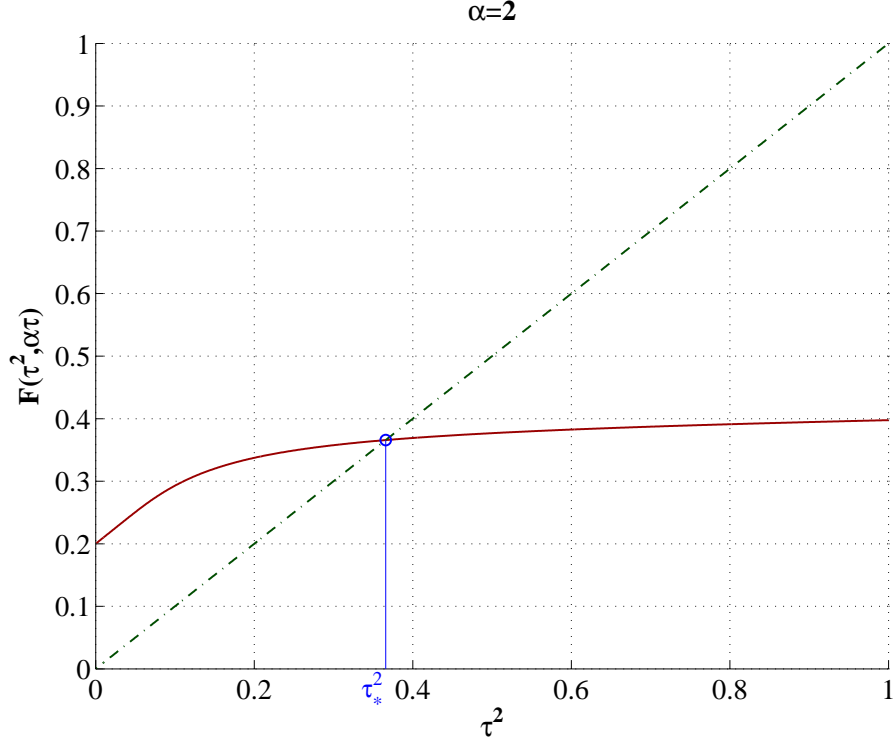


Figure 1: Mapping  $\tau^2 \mapsto F(\tau^2, \alpha\tau)$  for  $\alpha = 2$ ,  $\delta = 0.64$ ,  $\sigma^2 = 0.2$ ,  $p_{X_0}(\{+1\}) = p_{X_0}(\{-1\}) = 0.064$  and  $p_{X_0}(\{0\}) = 0.872$ .

**Proposition 1.3** ([DMM10]). *The function  $\alpha \mapsto \lambda(\alpha)$  is continuous on the interval  $(\alpha_{\min}, \infty)$  with  $\lambda(\alpha_{\min}+) = -\infty$  and  $\lim_{\alpha \rightarrow \infty} \lambda(\alpha) = \infty$ .*

*Therefore the function  $\lambda \mapsto \alpha(\lambda)$  satisfying Eq. (1.12) exists.*

A proof of this statement is provided in Section A.2. We will denote by  $\mathcal{A} = \alpha((0, \infty))$  the image of the function  $\alpha$ . Notice that the definition of  $\alpha$  is *a priori* not unique. We will see that uniqueness follows from our main theorem.

Examples of the mappings  $\tau^2 \mapsto F(\tau^2, \alpha\tau)$ ,  $\alpha \mapsto \tau_*(\alpha)$  and  $\alpha \mapsto \lambda(\alpha)$  are presented in Figures 1, 2, and 3 respectively.

We can now state our main result.

**Theorem 1.4.** *Let  $\{x_0(N), w(N), A(N)\}_{N \in \mathbb{N}}$  be a converging sequence of instances with the entries of  $A(N)$  iid normal with mean 0 and variance  $1/n$ . Denote by  $\hat{x}(\lambda; N)$  the LASSO estimator for instance  $(x_0(N), w(N), A(N))$ , with  $\sigma^2, \lambda > 0$ ,  $\mathbb{P}\{X_0 \neq 0\}$  and let  $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a pseudo-Lipschitz function. Then, almost surely*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(\hat{x}_i, x_{0,i}) = \mathbb{E} \left\{ \psi(\eta(X_0 + \tau_* Z; \theta_*), X_0) \right\}, \quad (1.12)$$

where  $Z \sim \mathcal{N}(0, 1)$  is independent of  $X_0 \sim p_{X_0}$ ,  $\tau_* = \tau_*(\alpha(\lambda))$  and  $\theta_* = \alpha(\lambda)\tau_*(\alpha(\lambda))$ .

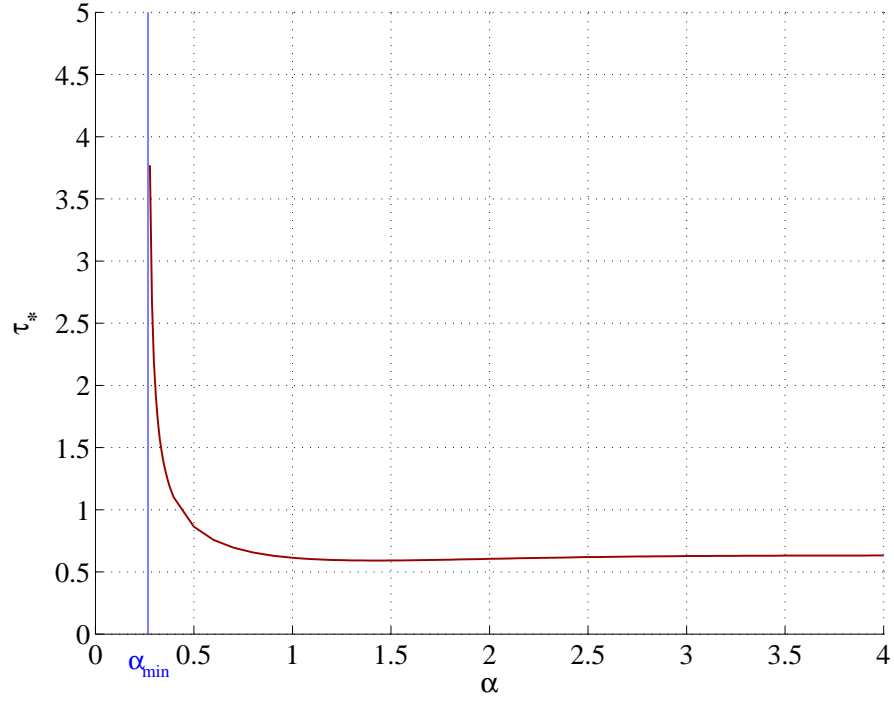


Figure 2: Mapping  $\alpha \mapsto \tau_*(\alpha)$  for the same parameters  $\delta, \sigma^2$  and distribution  $p_{X_0}$  as in Figure 1.

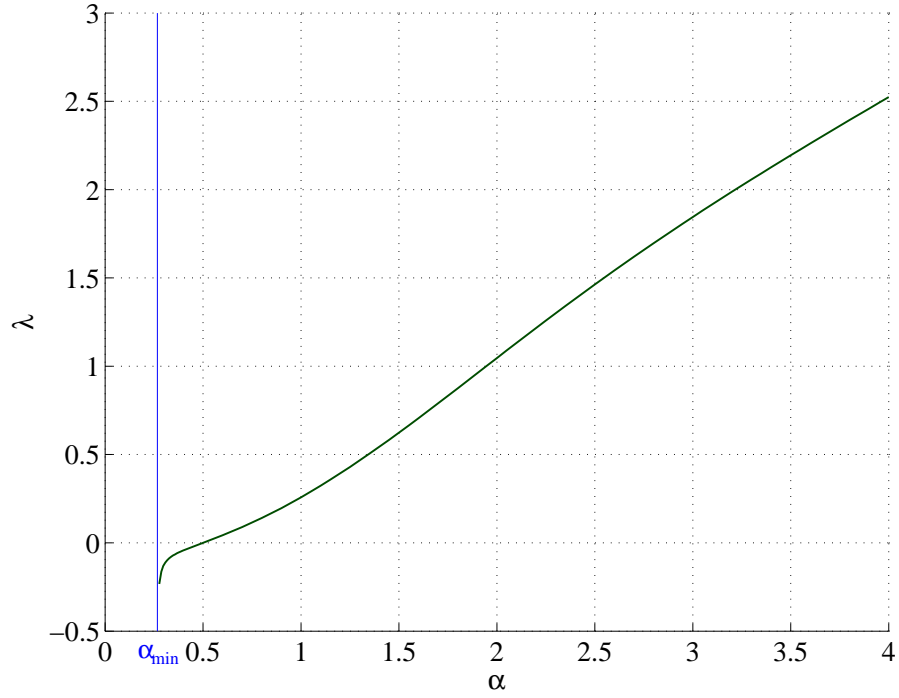


Figure 3: Mapping  $\alpha \mapsto \lambda(\alpha)$  for the same parameters  $\delta, \sigma^2$  and distribution  $p_{X_0}$  as in Figure 1.

As a corollary, the function  $\lambda \mapsto \alpha(\lambda)$  is indeed uniquely defined.

**Corollary 1.5.** *For any  $\lambda, \sigma^2 > 0$  there exists a unique  $\alpha > \alpha_{\min}$  such that  $\lambda(\alpha) = \lambda$  (with the function  $\alpha \rightarrow \lambda(\alpha)$  defined as in Eq. (1.11)).*

*Hence the function  $\lambda \mapsto \alpha(\lambda)$  is continuous non-decreasing with  $\alpha((0, \infty)) \equiv \mathcal{A} = (\alpha_0, \infty)$ .*

The proof of this corollary (which uses Theorem 1.4) is provided in Appendix A.3.

The assumption of a converging problem-sequence is important for the result to hold, while the hypothesis of gaussian measurement matrices  $A(N)$  is necessary for the proof technique to be correct. On the other hand, the restrictions  $\lambda, \sigma^2 > 0$ , and  $\mathbb{P}\{X_0 \neq 0\} > 0$  (whence  $\tau_* \neq 0$  using Eq. (1.11)) are made in order to avoid technical complications due to degenerate cases. Such cases can be resolved by continuity arguments.

The proof of Theorem 1.4 is given in Section 3.

### 1.3 Related work

The LASSO was introduced in [Tib96, CD95]. Several papers provide performance guarantees for the LASSO or similar convex optimization methods [CRT06, CT07], by proving upper bounds on the resulting mean square error. These works assume an appropriate ‘isometry’ condition to hold for  $A$ . While such condition hold with high probability for some random matrices, it is often difficult to verify them explicitly. Further, it is only applicable to very sparse vectors  $x_0$ . These restrictions are intrinsic to the worst-case point of view developed in [CRT06, CT07].

Guarantees have been proved for correct support recovery in [ZY06], under an appropriate ‘incoherence’ assumption on  $A$ . While support recovery is an interesting conceptualization for some applications (e.g. model selection), the metric considered in the present paper (mean square error) provides complementary information and is quite standard in many different fields.

Closer to the spirit of this paper [RFG09] derived expressions for the mean square error under the same model considered here. Similar results were presented recently in [KWT09, GBS09]. These papers argue that a sharp asymptotic characterization of the LASSO risk can provide valuable guidance in practical applications. For instance, it can be used to evaluate competing optimization methods on large scale applications, or to tune the regularization parameter  $\lambda$ .

Unfortunately, these results were non-rigorous and were obtained through the famously powerful ‘replica method’ from statistical physics [MM09].

Let us emphasize that the present paper offers two advantages over these recent developments: (i) It is completely *rigorous*, thus putting on a firmer basis this line of research; (ii) It is *algorithmic* in that the LASSO mean square error is shown to be equivalent to the one achieved by a low-complexity message passing algorithm.

## 2 Numerical illustrations

Theorem 1.4 assumes that the entries of matrix  $A$  are iid gaussians. We expect however the mean square error prediction to be robust and hold for much larger family of matrices. Rigorous evidence in this direction is presented in [KM10] where the normalized cost  $\mathcal{C}(\hat{x})/N$  is shown to have a limit as  $N \rightarrow \infty$  which is universal with respect to random matrices  $A$  with iid entries. (More precisely, it is universal provided  $\mathbb{E}\{A_{ij}\} = 0$ ,  $\mathbb{E}\{A_{ij}^2\} = 1/n$  and  $\mathbb{E}\{A_{ij}^6\} \leq C/n^3$  for some uniform constant  $C$ .)

Further, our result is asymptotic, while one might wonder how accurate it is for instances of moderate dimensions.

Numerical simulations were carried out in [DMM10, BBM10] and suggest that the result is robust and relevant already for  $N$  of the order of a few hundreds. As an illustration, we present in Figs. 4 and 5 the outcome of such simulations for two types of random matrices. Simulations with real data can be found in [BBM10]. We generated the signal vector randomly with entries in  $\{+1, 0, -1\}$  and  $\mathbb{P}(x_{0,i} = +1) = \mathbb{P}(x_{0,i} = -1) = 0.064$ . The noise vector  $w$  was generated by using i.i.d.  $\mathcal{N}(0, 0.2)$  entries.

We obtained the optimum estimator  $\hat{x}$  using **CVX**, a package for specifying and solving convex programs [GB10] and **OWLQN**, a package for solving large-scale versions of LASSO [AJ07]. We used several values of  $\lambda$  between 0 and 2 and  $N$  equal to 200, 500, 1000, and 2000. The aspect ratio of matrices was fixed in all cases to  $\delta = 0.64$ . For each case, the point  $(\lambda, \text{MSE})$  was plotted and the results are shown in the figures. Continuous lines corresponds to the asymptotic prediction by Theorem 1.4 for  $\psi(a, b) = (a - b)^2$ , namely

$$\lim_{N \rightarrow \infty} \frac{1}{N} \|\hat{x} - x_0\|^2 = \mathbb{E} \left\{ \left[ \eta(X_0 + \tau_* Z; \theta_*) - X_0 \right]^2 \right\} = \delta(\tau_*^2 - \sigma^2).$$

The agreement is remarkably good already for  $N, n$  of the order of a few hundreds, and deviations are consistent with statistical fluctuations.

The two figures correspond to different entries distributions: (i) Random gaussian matrices with aspect ratio  $\delta$  and iid  $\mathcal{N}(0, 1/n)$  entries (as in Theorem 1.4); (ii) Random  $\pm 1$  matrices with aspect ratio  $\delta$ . Each entry is independently equal to  $+1/\sqrt{n}$  or  $-1/\sqrt{n}$  with equal probability.

Notice that the asymptotic prediction has a minimum as a function of  $\lambda$ . The location of this minimum can be used to select the regularization parameter.

### 3 A structural property and proof of the main theorem

We will prove the following theorem which implies our main result, Theorem 1.4.

**Theorem 3.1.** *Assume the hypotheses of Theorem 1.4. Let  $\hat{x}(\lambda; N)$  the LASSO estimator for instance  $(x_0(N), w(N), A(N))$ , and denote by  $\{x^t(N)\}_{t \geq 0}$  the sequence of estimates produced by AMP. Then*

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \|x^t(N) - \hat{x}(\lambda; N)\|_2^2 = 0, \quad (3.1)$$

*almost surely.*

The rest of the paper is devoted to the proof of this theorem. Section 3.2 proves a structural property that is the key tool in this proof. Section 3.3 uses this property together with a few lemmas to prove Theorem 3.1

The proof of Theorem 1.4 follows immediately.



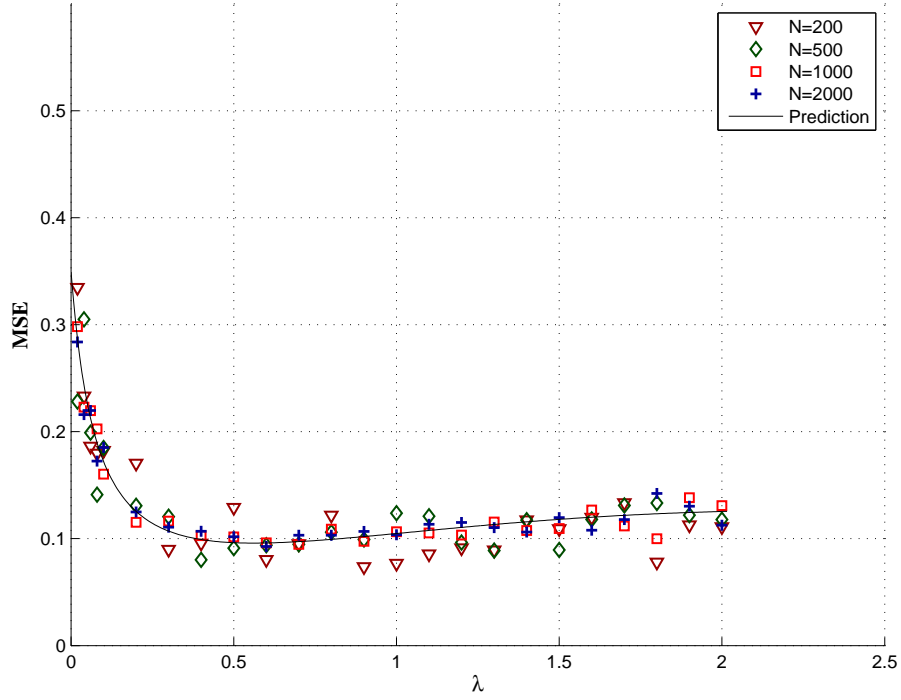


Figure 4: Mean square error (MSE) as a function of the regularization parameter  $\lambda$  compared to the asymptotic prediction for  $\delta = 0.64$  and  $\sigma^2 = 0.2$ . Here the measurement matrix  $A$  has iid  $N(0, 1/n)$  entries. Each point in this plot is generated by finding the LASSO predictor  $\hat{x}$  using a measurement vector  $y = Ax_0 + w$  for an independent signal vector  $x_0$ , an independent noise vector  $w$ , and an independent matrix  $A$ .

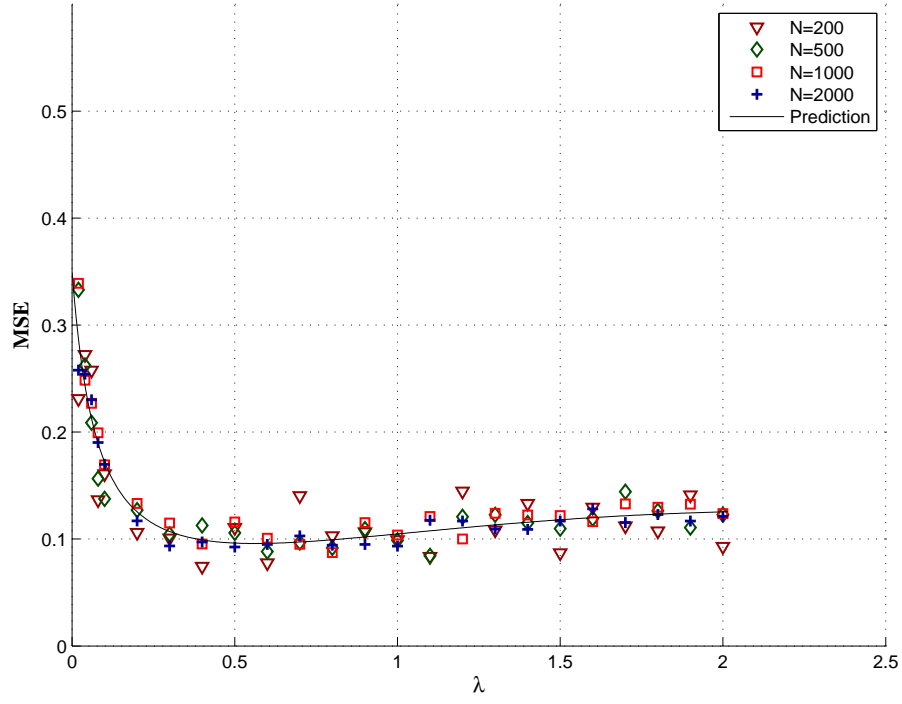


Figure 5: As in Fig. 4, but the measurement matrix  $A$  has iid entries that are equal to  $\pm 1/\sqrt{n}$  with equal probabilities.

*Proof of Theorem 1.4.* For any  $t \geq 0$ , we have, by the pseudo-Lipschitz property of  $\psi$ ,

$$\begin{aligned} \left| \frac{1}{N} \sum_{i=1}^N \psi(x_i^{t+1}, x_{0,i}) - \frac{1}{N} \sum_{i=1}^N \psi(\hat{x}_i, x_{0,i}) \right| &\leq \frac{L}{N} \sum_{i=1}^N |x_i^{t+1} - \hat{x}_i| (1 + 2|x_{0,i}| + |x_i^{t+1}| + |\hat{x}_i|) \\ &\leq \frac{L}{N} \|x^{t+1} - \hat{x}\|_2 \sqrt{\sum_{i=1}^N (1 + 2|x_{0,i}| + |x_i^{t+1}| + |\hat{x}_i|)^2} \\ &\leq L \frac{\|x^{t+1} - \hat{x}\|_2}{\sqrt{N}} \sqrt{4 + \frac{8\|x_0\|_2^2}{N} + \frac{4\|x^{t+1}\|_2^2}{N} + \frac{4\|\hat{x}\|_2^2}{N}}, \end{aligned}$$

where the second inequality follows by Cauchy-Schwarz. Next we take the limit  $N \rightarrow \infty$  followed by  $t \rightarrow \infty$ . The first term vanishes by Theorem 3.1. For the second term, note that  $\|x_0\|_2^2/N$  remains bounded since  $(x_0, w, A)$  is a converging sequence. The two terms  $\|x^{t+1}\|_2^2/N$  and  $\|\hat{x}\|_2^2/N$  also remain bounded in this limit because of state evolution (as proved in Lemma 3.3 below).

We then obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(\hat{x}_i, x_{0,i}) = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_i^{t+1}, x_{0,i}) = \mathbb{E} \left\{ \psi(\eta(X_0 + \tau_* Z; \theta_*), X_0) \right\},$$

where we used Theorem 1.1 and Proposition 1.2.  $\square$

### 3.1 Some notations

Before continuing, we introduce some useful notations. For any non-empty subset  $S$  of  $[m]$  and any  $k \times m$  matrix  $M$  we refer by  $M_S$  to the  $k$  by  $|S|$  sub-matrix of  $M$  that contains only the columns of  $M$  corresponding to  $S$ . The same notation is used for vectors  $v \in \mathbb{R}^m$ :  $v_S$  is the vector  $(v_i : i \in S)$ .

The transpose of matrix  $M$  is denoted by  $M^*$ .

We will often use the following scalar product for  $u, v \in \mathbb{R}^m$ :

$$\langle u, v \rangle \equiv \frac{1}{m} \sum_{i=1}^m u_i v_i. \quad (3.2)$$

Finally, the subgradient of a convex function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  at point  $x \in \mathbb{R}^m$  is denoted by  $\partial f(x)$ . In particular, remember that the subgradient of the  $\ell_1$  norm,  $x \mapsto \|x\|_1$  is given by

$$\partial \|x\|_1 = \{v \in \mathbb{R}^m \text{ such that } |v_i| \leq 1 \forall i \text{ and } x_i \neq 0 \Rightarrow v_i = \text{sign}(x_i)\}. \quad (3.3)$$

### 3.2 A structural property of the LASSO cost function

One main challenge in the proof of Theorem 1.4 lies in the fact that the function  $x \mapsto \mathcal{C}_{A,y}(x)$  is not –in general– strictly convex. Hence there can be, in principle, vectors  $x$  of cost very close to the optimum and nevertheless far from the optimum.

The following Lemma provides conditions under which this does not happen.

**Lemma 3.2.** *There exists a function  $\xi(\varepsilon, c_1, \dots, c_5)$  such that the following happens. If  $x, r \in \mathbb{R}^N$  satisfy the following conditions*

1.  $\|r\|_2 \leq c_1 \sqrt{N}$ ;
2.  $\mathcal{C}(x+r) \leq \mathcal{C}(x)$ ;
3. There exists  $\text{sg}(\mathcal{C}, x) \in \partial \mathcal{C}(x)$  with  $\|\text{sg}(\mathcal{C}, x)\|_2 \leq \sqrt{N} \varepsilon$ ;
4. Let  $v \equiv (1/\lambda)[A^*(y - Ax) + \text{sg}(\mathcal{C}, x)] \in \partial \|x\|_1$ , and  $S(c_2) \equiv \{i \in [N] : |v_i| \geq 1 - c_2\}$ . Then, for any  $S' \subseteq [N]$ ,  $|S'| \leq c_3 N$ , we have  $\sigma_{\min}(A_{S(c_2) \cup S'}) \geq c_4$ ;
5. The maximum and minimum non-zero singular value of  $A$  satisfy  $c_5^{-1} \leq \sigma_{\min}(A)^2 \leq \sigma_{\max}(A)^2 \leq c_5$ .

Then  $\|r\|_2 \leq \sqrt{N} \xi(\varepsilon, c_1, \dots, c_5)$ . Further for any  $c_1, \dots, c_5 > 0$ ,  $\xi(\varepsilon, c_1, \dots, c_5) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Further, if  $\ker(A) = \{0\}$ , the same conclusion holds under assumptions 1, 2, 3, 5.

*Proof.* Throughout the proof we denote  $\xi_1, \xi_2, \dots$  functions of the constants  $c_1, \dots, c_5 > 0$  and of  $\varepsilon$  such that  $\xi_i(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (we shall omit the dependence of  $\xi_i$  on  $\varepsilon$ ).

Let  $S = \text{supp}(x) \subseteq [N]$ . We have

$$\begin{aligned}
0 & \stackrel{(a)}{\geq} \left( \frac{\mathcal{C}(x+r) - \mathcal{C}(x)}{N} \right) \\
& \stackrel{(b)}{=} \lambda \left( \frac{\|x_S + r_S\|_1 - \|x_S\|_1}{N} \right) + \frac{\lambda \|r_{\bar{S}}\|_1 + \frac{1}{2} \|y - Ax - Ar\|_2^2 - \frac{1}{2} \|y - Ax\|_2^2}{N} \\
& \stackrel{(c)}{=} \lambda \left( \frac{\|x_S + r_S\|_1 - \|x_S\|_1}{N} - \langle \text{sign}(x_S), r_S \rangle \right) + \lambda \left( \frac{\|r_{\bar{S}}\|_1}{N} - \langle v_{\bar{S}}, r_{\bar{S}} \rangle \right) + \lambda \langle v, r \rangle - \langle y - Ax, Ar \rangle + \frac{\|Ar\|_2^2}{2N} \\
& \stackrel{(d)}{=} \lambda \left( \frac{\|x_S + r_S\|_1 - \|x_S\|_1}{N} - \langle \text{sign}(x_S), r_S \rangle \right) + \lambda \left( \frac{\|r_{\bar{S}}\|_1}{N} - \langle v_{\bar{S}}, r_{\bar{S}} \rangle \right) + \langle \text{sg}(\mathcal{C}, x), r \rangle + \frac{\|Ar\|_2^2}{2N},
\end{aligned}$$

where (a) follows from hypothesis (2), (c) from the fact that  $v_S = \text{sign}(x_S)$  since  $v \in \partial \|x\|_1$ , and (d) from the definition of  $(v)$ . Using hypothesis (1) and (3), we get by Cauchy-Schwarz

$$\lambda \left( \frac{\|x_S + r_S\|_1 - \|x_S\|_1}{N} - \langle \text{sign}(x_S), r_S \rangle \right) + \lambda \left( \frac{\|r_{\bar{S}}\|_1}{N} - \langle v_{\bar{S}}, r_{\bar{S}} \rangle \right) + \frac{\|Ar\|_2^2}{2N} \leq c_1 \varepsilon.$$

Since each of the three terms on the left-hand side is non-negative it follows that

$$\frac{\|r_{\bar{S}}\|_1}{N} - \langle v_{\bar{S}}, r_{\bar{S}} \rangle \leq \xi_1(\varepsilon), \tag{3.4}$$

$$\|Ar\|_2^2 \leq N \xi_1(\varepsilon). \tag{3.5}$$

Write  $r = r^\perp + r^\parallel$ , with  $r^\parallel \in \ker(A)$  and  $r^\perp \perp \ker(A)$ . It follows from Eq. (3.5) and hypothesis (5) that

$$\|r^\perp\|_2^2 \leq N c_5 \xi_1(\varepsilon). \tag{3.6}$$

In the case  $\ker(A) = \{0\}$ , the proof is concluded. In the case  $\ker(A) \neq \{0\}$ , we need to prove an analogous bound for  $r^\parallel$ . From Eq. (3.4) together with  $\|r_{\bar{S}}^\perp\|_1 \leq \sqrt{N} \|r_{\bar{S}}^\perp\|_2 \leq \sqrt{N} \|r^\perp\|_2 \leq N \sqrt{c_5 \xi_1(\varepsilon)}$ , we get

$$Ar^\parallel = 0, \tag{3.7}$$

$$\frac{\|r_{\bar{S}}^\parallel\|_1}{N} - \langle v_{\bar{S}}, r_{\bar{S}}^\parallel \rangle \leq \xi_2(\varepsilon), \tag{3.8}$$

Notice that  $\overline{S}(c_2) \subseteq \overline{S}$ . From Eq. (3.8) and definition of  $S(c_2)$  it follows that

$$\|r_{\overline{S}(c_2)}^\parallel\|_1 \leq \frac{\|r_{\overline{S}(c_2)}^\parallel\|_1 - N \langle v_{\overline{S}(c_2)}, r_{\overline{S}(c_2)}^\parallel \rangle}{c_2} \quad (3.9)$$

$$\leq N c_2^{-1} \xi_2(\varepsilon). \quad (3.10)$$

Let us first consider the case  $|\overline{S}(c_2)| \geq N c_3/2$ . Then partition  $\overline{S}(c_2) = \cup_{\ell=1}^K S_\ell$ , where  $(N c_3/2) \leq |S_\ell| \leq N c_3$ , and for each  $i \in S_\ell$ ,  $j \in S_{\ell+1}$ ,  $|r_i^\parallel| \geq |r_j^\parallel|$ . Also define  $\overline{S}_+ \equiv \cup_{\ell=2}^K S_\ell \subseteq \overline{S}(c_2)$ . Since, for any  $i \in S_\ell$   $|r_i^\parallel| \leq \|r_{S_{\ell-1}}^\parallel\|_1/|S_{\ell-1}|$ , we have

$$\begin{aligned} \|r_{\overline{S}_+}^\parallel\|_2^2 &= \sum_{\ell=2}^K \|r_{S_\ell}^\parallel\|_2^2 \leq \sum_{\ell=2}^K |S_\ell| \left( \frac{\|r_{S_{\ell-1}}^\parallel\|_1}{|S_{\ell-1}|} \right)^2 \\ &\leq \frac{4}{N c_3} \sum_{\ell=2}^K \|r_{S_{\ell-1}}^\parallel\|_1^2 \leq \frac{4}{N c_3} \left( \sum_{\ell=2}^K \|r_{S_{\ell-1}}^\parallel\|_1 \right)^2 \\ &\leq \frac{4}{N c_3} \|r_{\overline{S}(c_2)}^\parallel\|_1^2 \leq \frac{4 \xi_2(\varepsilon)^2}{c_2^2 c_3} N \equiv N \xi_3(\varepsilon). \end{aligned}$$

To conclude the proof, it is sufficient to prove an analogous bound for  $\|r_{S_+}^\parallel\|_2^2$  with  $S_+ = [N] \setminus \overline{S}_+ = S(c_2) \cup S_1$ . Since  $|S_1| \leq N c_3$ , we have by hypothesis (4) that  $\sigma_{\min}(A_{S_+}) \geq c_4$ . Since  $0 = A r^\parallel = A_{S_+} r_{S_+}^\parallel + A_{\overline{S}_+} r_{\overline{S}_+}^\parallel$ , we have

$$c_4^2 \|r_{S_+}^\parallel\|_2^2 \leq \|A_{S_+} r_{S_+}^\parallel\|_2^2 = \|A_{\overline{S}_+} r_{\overline{S}_+}^\parallel\|_2^2 \leq c_5 \|r_{\overline{S}_+}^\parallel\|_2^2 \leq c_5 N \xi_3(\varepsilon).$$

This finishes the proof when  $|\overline{S}(c_2)| \geq N c_3/2$ . Note that if this assumption does not hold then we have  $\overline{S}_+ = \emptyset$  and  $S_+ = [N]$ . Hence, the result follows as a special case of above.  $\square$

### 3.3 Proof of Theorem 3.1

The proof is based on a series of Lemmas that are used to check the assumptions of Lemma 3.2

The first one is an upper bound on the  $\ell_2$ -norm of AMP estimates, and of the LASSO estimate. Its proof is deferred to Section 5.1.

**Lemma 3.3.** *Under the conditions of Theorem 1.4, assume  $\lambda > 0$  and  $\alpha = \alpha(\lambda)$ . Denote by  $\widehat{x}(\lambda; N)$  the LASSO estimator and by  $\{x^t(N)\}$  the sequence of AMP estimates. Then there is a constant  $B$  such that for all  $t \geq 0$ , almost surely*

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \langle x^t(N), x^t(N) \rangle < B, \quad (3.11)$$

$$\lim_{N \rightarrow \infty} \langle \widehat{x}(\lambda; N), \widehat{x}(\lambda; N) \rangle < B. \quad (3.12)$$

The second Lemma implies that the estimates of AMP are approximate minima, in the sense that the cost function  $\mathcal{C}$  admits a small subgradient at  $x^t$ , when  $t$  is large. The proof is deferred to Section 5.2.

**Lemma 3.4.** *Under the conditions of Theorem 1.4, for all  $t$  there exists a subgradient  $\text{sg}(\mathcal{C}, x^t)$  of  $\mathcal{C}$  at point  $x^t$  such that almost surely,*

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \|\text{sg}(\mathcal{C}, x^t)\|^2 = 0. \quad (3.13)$$

The next lemma implies that submatrices of  $A$  constructed using the first  $t$  iterations of the AMP algorithm are non-singular (more precisely, have singular values bounded away from 0). The proof can be found in Section 5.3.

**Lemma 3.5.** *Let  $S \subseteq [N]$  be measurable on the  $\sigma$ -algebra  $\mathfrak{S}_t$  generated by  $\{z^0, \dots, z^{t-1}\}$  and  $\{x^0 + A^* z^0, \dots, x^{t-1} + A^* z^{t-1}\}$  and assume  $|S| \leq N(\delta - c)$  for some  $c > 0$ . Then there exists  $a_1 = a_1(c) > 0$  (independent of  $t$ ) and  $a_2 = a_2(c, t) > 0$  (depending on  $t$  and  $c$ ) such that*

$$\min_{S'} \{ \sigma_{\min}(A_{S \cup S'}) : S' \subseteq [N], |S'| \leq a_1 N \} \geq a_2, \quad (3.14)$$

with probability converging to 1 as  $N \rightarrow \infty$ .

We will apply this lemma to a specific choice of the set  $S$ . Namely, defining

$$v^t \equiv \frac{1}{\theta_{t-1}} (x^{t-1} + A^* z^{t-1} - x^t), \quad (3.15)$$

we will then consider the set

$$S_t(\gamma) \equiv \{ i \in [N] : |v_i^t| \geq 1 - \gamma \}, \quad (3.16)$$

for  $\gamma \in (0, 1)$ . Our last lemma shows that this sequence of sets  $S_t(\gamma)$  ‘converges’ in the following sense. The proof can be found in Section 5.4.

**Lemma 3.6.** *Fix  $\gamma \in (0, 1)$  and let the sequence  $\{S_t(\gamma)\}_{t \geq 0}$  be defined as in Eq. (3.16) above. For any  $\xi > 0$  there exists  $t_* = t_*(\xi, \gamma) < \infty$  such that, for all  $t_2 \geq t_1 \geq t_*$*

$$\lim_{N \rightarrow \infty} \mathbb{P}\{|S_{t_2}(\gamma) \setminus S_{t_1}(\gamma)| \geq N\xi\} = 0. \quad (3.17)$$

The last two lemmas imply the following.

**Proposition 3.7.** *There exist constants  $\gamma_1 \in (0, 1)$ ,  $\gamma_2, \gamma_3 > 0$  and  $t_{\min} < \infty$  such that, for any  $t \geq t_{\min}$ ,*

$$\min \{ \sigma_{\min}(A_{S_t(\gamma_1) \cup S'}) : S' \subseteq [N], |S'| \leq \gamma_2 N \} \geq \gamma_3 \quad (3.18)$$

with probability converging to 1 as  $N \rightarrow \infty$ .

*Proof.* First notice that, for any fixed  $\gamma$ , the set  $S_t(\gamma)$  is measurable on  $\mathfrak{S}_t$ . Indeed by Eq. (1.5)  $\mathfrak{S}_t$  contains  $\{x^0, \dots, x^t\}$  as well, and hence it contains  $v^t$  which is a linear combination of  $x^{t-1} + A^* z^{t-1}$ ,  $x^t$ . Finally  $S_t(\gamma)$  is obviously a measurable function of  $v^t$ .

Using Lemma F.3(b) the empirical distribution of  $(x_0 - A^*z^{t-1} - x^{t-1}, x_0)$  converges weakly to  $(\tau_{t-1}Z, X_0)$  for  $Z \sim N(0, 1)$  independent of  $X_0 \sim p_{X_0}$ . (Following the notation of [BM10], we let  $h^t = x_0 - A^*z^{t-1} - x^{t-1}$ .) Therefore, for any constant  $\gamma$  we have almost surely

$$\lim_{N \rightarrow \infty} \frac{|S_t(\gamma)|}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\left\{ \frac{1}{\theta_{t-1}} |x_i^{t-1} + [A^*z^{t-1}]_i - x_i^t| \geq 1 - \gamma \right\}} \quad (3.19)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\left\{ \frac{1}{\theta_{t-1}} |x_0 - h^t - \eta(x_0 - h^t, \theta_{t-1})| \geq 1 - \gamma \right\}} \quad (3.20)$$

$$= \mathbb{P} \left\{ \frac{1}{\theta_{t-1}} |X_0 + \tau_{t-1}Z - \eta(X_0 + \tau_{t-1}Z, \theta_{t-1})| \geq 1 - \gamma \right\}. \quad (3.21)$$

The last equality follows from the weak convergence of the empirical distribution of  $\{(h_i, x_{0,i})\}_{i \in [N]}$  (from Lemma F.3(b), which takes the same form as Theorem 3.1), together with the absolute continuity of the distribution of  $|X_0 + \tau_{t-1}Z - \eta(X_0 + \tau_{t-1}Z, \theta_{t-1})|$ .

Now, combining

$$\left| X_0 + \tau_{t-1}Z - \eta(X_0 + \tau_{t-1}Z, \theta_{t-1}) \right| = \begin{cases} \theta_{t-1} & \text{When } |X_0 + \tau_{t-1}Z| \geq \theta_{t-1}, \\ |X_0 + \tau_{t-1}Z| & \text{Otherwise,} \end{cases}$$

and Eq. (3.21) we obtain almost surely

$$\lim_{N \rightarrow \infty} \frac{|S_t(\gamma)|}{N} = \mathbb{E} \left\{ \eta'(X_0 + \tau_{t-1}Z, \theta_{t-1}) \right\} + \mathbb{P} \left\{ (1 - \gamma) \leq \frac{1}{\theta_{t-1}} |X_0 + \tau_{t-1}Z| \leq 1 \right\}. \quad (3.22)$$

It is easy to see that the second term  $\mathbb{P} \{ 1 - \gamma \leq (1/\theta_{t-1}) |X + \tau_{t-1}Z| \leq 1 \}$  converges to 0 as  $\gamma \rightarrow 0$ . On the other hand, using Eq. (1.11) and the fact that  $\lambda(\alpha) > 0$  the first term will be strictly smaller than  $\delta$  for large enough  $t$ . Hence, we can choose constants  $\gamma_1 \in (0, 1)$  and  $c > 0$  such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \{ |S_t(\gamma_1)| < N(\delta - c) \} = 1. \quad (3.23)$$

for all  $t$  larger than some  $t_{\min,1}(c)$ .

For any  $t \geq t_{\min,1}(c)$  we can apply Lemma 3.5 for some  $a_1(c)$ ,  $a_2(c, t) > 0$ . Fix  $c > 0$  and let  $a_1 = a_1(c)$  be fixed as well. Let  $t_{\min} = \max(t_{\min,1}, t_*(a_1/2, \gamma_1))$  (with  $t_*(\cdot)$  defined as per Lemma 3.6). Take  $a_2 = a_2(c, t_{\min})$ . Obviously  $t \mapsto a_2(c, t)$  is non-increasing. Then we have, by Lemma 3.5

$$\min \{ \sigma_{\min}(A_{S_{t_{\min}}(\gamma_1) \cup S'}) : |S'| \leq a_1 N \} \geq a_2, \quad (3.24)$$

and by Lemma 3.6

$$|S_t(\gamma_1) \setminus S_{t_{\min}}(\gamma_1)| \leq N a_1 / 2, \quad (3.25)$$

where both events hold with probability converging to 1 as  $N \rightarrow \infty$ . The claim follows with  $\gamma_2 = a_1(c)/2$  and  $\gamma_3 = a_2(c, t_{\min})$ .  $\square$

We are now in position to prove Theorem 3.1.

*Proof of Theorem 3.1.* We apply Lemma 3.2 to  $x = x^t$ , the AMP estimate and  $r = \hat{x} - x^t$  the distance from the LASSO optimum. The thesis follows by checking conditions 1–5. Namely we need to show that there exists constants  $c_1, \dots, c_5 > 0$  and, for each  $\varepsilon > 0$  some  $t = t(\varepsilon)$  such that 1–5 hold with probability going to 1 as  $N \rightarrow \infty$ .

*Condition 1* holds by Lemma 3.3.

*Condition 2* is immediate since  $x + r = \hat{x}$  minimizes  $\mathcal{C}(\cdot)$ .

*Condition 3* follows from Lemma 3.4 with  $\varepsilon$  arbitrarily small for  $t$  large enough.

*Condition 4.* Notice that this condition only needs to be verified for  $\delta < 1$ .

Take  $v = v^t$  as defined in Eq. (3.15). Using the definition (1.5), it is easy to check that  $|v_i^t| \leq 1$  if  $x_i^t = 0$  and  $v_i^t = \text{sign}(x_i^t)$  otherwise. In other words  $v^t \in \partial\|x\|_1$  as required. Further by inspection of the proof of Lemma 3.4, it follows that  $v^t = (1/\lambda)[A^*(y - Ax^t) + \text{sg}(\mathcal{C}, x^t)]$ , with  $\text{sg}(\mathcal{C}, x^t)$  the subgradient bounded in that lemma (cf. Eq. (5.3)). The condition then holds by Proposition 3.7.

*Condition 5* follows from standard limit theorems on the singular values of Wishart matrices (cf. Theorem F.2).  $\square$

## 4 State evolution estimates

This section contains a reminder of the state-evolution method developed in [BM10]. We also state some extensions of those results that will be proved in the appendices.

### 4.1 State evolution

AMP, cf. Eq. (1.5) is a special case of the general iterative procedure given by Eq. (3.1) of [BM10]. This takes the general form

$$\begin{aligned} h^{t+1} &= A^* m^t - \xi_t q^t, & m^t &= g_t(b^t, w), \\ b^t &= A q^t - \lambda_t m^{t-1}, & q^t &= f_t(h^t, x_0), \end{aligned} \quad (4.1)$$

where  $\xi_t = \langle g'(b^t, w) \rangle$ ,  $\lambda_t = \frac{1}{\delta} \langle f'_t(h^t, x_0) \rangle$  (both derivatives are with respect to the first argument).

This reduction can be seen by defining

$$h^{t+1} = x_0 - (A^* z^t + x^t), \quad (4.2)$$

$$q^t = x^t - x_0, \quad (4.3)$$

$$b^t = w - z^t, \quad (4.4)$$

$$m^t = -z^t, \quad (4.5)$$

where

$$f_t(s, x_0) = \eta_{t-1}(x_0 - s) - x_0, \quad g_t(s, w) = s - w, \quad (4.6)$$

and the initial condition is  $q^0 = -x_0$ .



Regarding  $h^t, b^t$  as column vectors, the equations for  $b^0, \dots, b^{t-1}$  and  $h^1, \dots, h^t$  can be written in matrix form as:

$$\underbrace{[h^1 + \xi_0 q^0 | h^2 + \xi_1 q^1 | \dots | h^t + \xi_{t-1} q^{t-1}]}_{X_t} = A^* \underbrace{[m^0 | \dots | m^{t-1}]}_{M_t}, \quad (4.7)$$

$$\underbrace{[b^0 | b^1 + \lambda_1 m^0 | \dots | b^{t-1} + \lambda_{t-1} m^{t-2}]}_{Y_t} = A \underbrace{[q^0 | \dots | q^{t-1}]}_{Q_t}. \quad (4.8)$$

or in short  $Y_t = A Q_t$  and  $X_t = A^* M_t$ .

Following [BM10], we define  $\mathfrak{S}_t$  as the  $\sigma$ -algebra generated by  $b^0, \dots, b^{t-1}, m^0, \dots, m^{t-1}, h^1, \dots, h^t$ , and  $q^0, \dots, q^t$ . The conditional distribution of the random matrix  $A$  given the  $\sigma$ -algebra  $\mathfrak{S}_t$ , is given by

$$A|_{\mathfrak{S}_t} \stackrel{d}{=} E_t + \mathcal{P}_t(\tilde{A}). \quad (4.9)$$

Here  $\tilde{A} \stackrel{d}{=} A$  is a random matrix independent of  $\mathfrak{S}_t$ , and  $E_t = \mathbb{E}(A|_{\mathfrak{S}_t})$  is given by

$$E_t = Y_t(Q_t^* Q_t)^{-1} Q_t^* + M_t(M_t^* M_t)^{-1} X_t^* - M_t(M_t^* M_t)^{-1} M_t^* Y_t(Q_t^* Q_t)^{-1} Q_t^*. \quad (4.10)$$

Further,  $\mathcal{P}_t$  is the orthogonal projector onto subspace  $V_t = \{A | A Q_t = 0, A^* M_t = 0\}$ , defined by

$$\mathcal{P}_t(\tilde{A}) = P_{M_t}^\perp \tilde{A} P_{Q_t}^\perp.$$

Here  $P_{M_t}^\perp = I - P_{M_t}$ ,  $P_{Q_t}^\perp = I - P_{Q_t}$ , and  $P_{Q_t}$ ,  $P_{M_t}$  are orthogonal projector onto column spaces of  $Q_t$  and  $M_t$  respectively.

Before proceeding, it is convenient to introduce the notation

$$\omega_t \equiv \frac{1}{\delta} \langle \eta'(A^* z^{t-1} + x^{t-1}; \theta_{t-1}) \rangle$$

to denote the coefficient of  $z^{t-1}$  in Eq. (1.5). Using  $h^t = x_0 - A^* z^{t-1} - x^{t-1}$  and Lemma F.3(b) (proved in [BM10]) we get, almost surely,

$$\lim_{N \rightarrow \infty} \omega_t = \omega_t^\infty \equiv \frac{1}{\delta} \mathbb{E}[\eta'(X_0 + \tau_{t-1} Z; \theta_{t-1})]. \quad (4.11)$$

Notice that the function  $\eta'(\cdot; \theta_{t-1})$  is discontinuous and therefore Lemma F.3(b) does not apply immediately. On the other hand, this implies that the empirical distribution of  $\{(A^* z_i^{t-1} + x_i^{t-1}, x_{0,i})\}_{1 \leq i \leq N}$  converges weakly to the distribution of  $(X_0 + \tau_{t-1} Z, X_0)$ . The claim follows from the fact that  $X_0 + \tau_{t-1} Z$  has a density, together with the standard properties of weak convergence.

## 4.2 Some consequences and generalizations

We begin with a simple calculation, that will be useful.

**Lemma 4.1.** *If  $\{z^t\}_{t \geq 0}$  are the AMP residuals, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|z^t\|^2 = \tau_t^2. \quad (4.12)$$

*Proof.* Using representation (4.5) and Lemma F.3(b)(c), we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|z^t\|^2 \stackrel{\text{a.s.}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \|m^t\|^2 \stackrel{\text{a.s.}}{=} \lim_{N \rightarrow \infty} \frac{1}{N} \|h^{t+1}\|^2 = \tau_t^2.$$

□

Next, we need to generalize state evolution to compute large system limits for functions of  $x^t$ ,  $x^s$ , with  $t \neq s$ . To this purpose, we define the covariances  $\{R_{s,t}\}_{s,t \geq 0}$  recursively by

$$R_{s+1,t+1} = \sigma^2 + \frac{1}{\delta} \mathbb{E} \left\{ [\eta(X_0 + Z_s; \theta_s) - X_0] [\eta(X_0 + Z_t; \theta_t) - X_0] \right\}, \quad (4.13)$$

with  $(Z_s, Z_t)$  jointly gaussian, independent from  $X_0 \sim p_{X_0}$  with zero mean and covariance given by  $\mathbb{E}\{Z_s^2\} = R_{s,s}$ ,  $\mathbb{E}\{Z_t^2\} = R_{t,t}$ ,  $\mathbb{E}\{Z_s Z_t\} = R_{s,t}$ . The boundary condition is fixed by letting  $R_{0,0} = \sigma^2 + \mathbb{E}\{X_0^2\}/\delta$  and

$$R_{0,t+1} = \sigma^2 + \frac{1}{\delta} \mathbb{E} \left\{ [\eta(X_0 + Z_t; \theta_t) - X_0] (-X_0) \right\}, \quad (4.14)$$

with  $Z_t \sim N(0, R_{t,t})$  independent of  $X_0$ . This determines by the above recursion  $R_{t,s}$  for all  $t \geq 0$  and for all  $s \geq 0$ .

With these definition, we have the following generalization of Theorem 1.1.

**Theorem 4.2.** *Let  $\{x_0(N), w(N), A(N)\}_{N \in \mathbb{N}}$  be a converging sequence of instances with the entries of  $A(N)$  iid normal with mean 0 and variance  $1/n$  and let  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a pseudo-Lipschitz function. Then, for all  $s \geq 0$  and  $t \geq 0$  almost surely*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_i^s + (A^* z^s)_i, x_i^t + (A^* z^t)_i, x_{0,i}) = \mathbb{E} \left\{ \psi(X_0 + Z_s, X_0 + Z_t, X_0) \right\}, \quad (4.15)$$

where  $(Z_s, Z_t)$  jointly gaussian, independent from  $X_0 \sim p_{X_0}$  with zero mean and covariance given by  $\mathbb{E}\{Z_s^2\} = R_{s,s}$ ,  $\mathbb{E}\{Z_t^2\} = R_{t,t}$ ,  $\mathbb{E}\{Z_s Z_t\} = R_{s,t}$ .

Notice that the above implies in particular, for any pseudo-Lipschitz function  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_i^{s+1}, x_i^{t+1}, x_{0,i}) = \mathbb{E} \left\{ \psi(\eta(X_0 + Z_s; \theta_s), \eta(X_0 + Z_t; \theta_t), X_0) \right\}. \quad (4.16)$$

Clearly this result reduces to Theorem 1.1 in the case  $s = t$  by noting that  $R_{t,t} = \tau_t^2$ . The general proof can be found in Appendix B.

The following lemma implies that, asymptotically for large  $N$ , the AMP estimates converge.

**Lemma 4.3.** *Under the condition of Theorem 1.4, the estimates  $\{x^t\}_{t \geq 0}$  and residuals  $\{z^t\}_{t \geq 0}$  of AMP almost surely satisfy*

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \|x^t - x^{t-1}\|^2 = 0, \quad \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N} \|z^t - z^{t-1}\|^2 = 0. \quad (4.17)$$

The proof is deferred to Appendix C.

## 5 Proofs of auxiliary lemmas

### 5.1 Proof of Lemma 3.3

In order to bound the norm of  $x^t$ , we use state evolution, Theorem 1.1, for the function  $\psi(a, b) = a^2$ ,

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \langle x^t, x^t \rangle \stackrel{\text{a.s.}}{=} \mathbb{E} \{ \eta(X_0 + \tau_* Z; \theta_*)^2 \}$$

for  $Z \sim \mathcal{N}(0, 1)$  and independent of  $X_0 \sim p_{X_0}$ . The expectation on the right hand side is bounded and hence  $\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \langle x^t, x^t \rangle$  is bounded.

For  $\hat{x}$ , first note that

$$\begin{aligned} \frac{1}{N} \mathcal{C}(\hat{x}) &\leq \frac{1}{N} \mathcal{C}(0) = \frac{1}{2N} \|y\|^2 \\ &= \frac{1}{2N} \|Ax_0 + w\|^2 \\ &\leq \frac{\|w\|^2 + \sigma_{\max}(A)^2 \|x_0\|^2}{2N} \leq B_1. \end{aligned} \tag{5.1}$$

The last bound holds almost surely as  $N \rightarrow \infty$ , using standard asymptotic estimate on the singular values of random matrices (cf. Theorem F.2) implying that  $\sigma_{\max}(A)$  has a bounded limit almost surely, together with the fact that  $(x_0, w, A)$  is a converging sequence.

Now, decompose  $\hat{x}$  as  $\hat{x} = \hat{x}_{\parallel} + \hat{x}_{\perp}$  where  $\hat{x}_{\parallel} \in \ker(A)$  and  $\hat{x}_{\perp} \in \ker(A)^{\perp}$  (the orthogonal complement of  $\ker(A)$ ). Since,  $\hat{x}_{\parallel}$  belongs to the random subspace  $\ker(A)$  with dimension  $N - n = N(1 - \delta)$ , Kashin theorem (cf. Theorem F.1) implies that there exists a positive constant  $c_1 = c_1(\delta)$  such that

$$\begin{aligned} \frac{1}{N} \|\hat{x}\|^2 &= \frac{1}{N} \|\hat{x}_{\parallel}\|^2 + \frac{1}{N} \|\hat{x}_{\perp}\|^2 \\ &\leq c_1 \left( \frac{\|\hat{x}_{\parallel}\|_1}{N} \right)^2 + \frac{1}{N} \|\hat{x}_{\perp}\|^2. \end{aligned}$$

Hence, by using triangle inequality and Cauchy-Schwarz, we get

$$\begin{aligned} \frac{1}{N} \|\hat{x}\|^2 &\leq 2c_1 \left( \frac{\|\hat{x}_{\parallel}\|_1}{N} \right)^2 + 2c_1 \left( \frac{\|\hat{x}_{\perp}\|_1}{N} \right)^2 + \frac{1}{N} \|\hat{x}_{\perp}\|^2 \\ &\leq 2c_1 \left( \frac{\|\hat{x}_{\parallel}\|_1}{N} \right)^2 + \frac{2c_1 + 1}{N} \|\hat{x}_{\perp}\|^2. \end{aligned}$$

By definition of cost function we have  $\|\hat{x}\|_1 \leq \lambda^{-1} \mathcal{C}(\hat{x})$ . Further, limit theorems for the eigenvalues of Wishart matrices (cf. Theorem F.2) imply that there exists a constant  $c = c(\delta)$  such that asymptotically almost surely  $\|\hat{x}_{\perp}\|^2 \leq c \|A\hat{x}_{\perp}\|^2$ . Therefore (denoting by  $c_i : i = 2, 3, 4$  bounded constants), we have

$$\begin{aligned} \frac{1}{N} \|\hat{x}\|^2 &\leq 2c_1 \left( \frac{\|\hat{x}_{\parallel}\|_1}{N} \right)^2 + \frac{c_2}{N} \|A\hat{x}_{\perp}\|^2 \\ &\leq 2c_1 \left( \frac{\|\hat{x}_{\parallel}\|_1}{N} \right)^2 + \frac{2c_2}{N} \|y - A\hat{x}_{\perp}\|^2 + \frac{2c_2}{N} \|y\|^2 \\ &\leq c_3 \left( \frac{\mathcal{C}(\hat{x})}{N} \right)^2 + 2c_2 \frac{\mathcal{C}(\hat{x})}{N} + \frac{2c_2}{N} \|Ax_0 + w\|^2. \end{aligned}$$

The claim follows by using the Eq. (5.1) to bound  $\mathcal{C}(\hat{x})/N$  and using  $\|Ax_0 + w\|^2 \leq \sigma_{\max}(A)^2 \|x_0\|^2 + \|w\|^2 \leq 2NB_1$  to bound the last term.  $\square$

## 5.2 Proof of Lemma 3.4

First note that equation  $x^t = \eta(A^*z^{t-1} + x^{t-1}; \theta_{t-1})$  of AMP implies

$$\begin{aligned} x_i^t + \theta_{t-1} \text{sign}(x_i^t) &= [A^*z^{t-1}]_i + x_i^{t-1}, & \text{if } x_i^t \neq 0, \\ \left| [A^*z^{t-1}]_i + x_i^{t-1} \right| &\leq \theta_{t-1}, & \text{if } x_i^t = 0. \end{aligned} \quad (5.2)$$

Therefore, the vector  $\text{sg}(\mathcal{C}, x^t) \equiv \lambda s^t - A^*(y - Ax^t)$  where

$$s_i^t = \begin{cases} \text{sign}(x_i^t) & \text{if } x_i^t \neq 0, \\ \frac{1}{\theta_{t-1}} \left\{ [A^*z^{t-1}]_i + x_i^{t-1} \right\} & \text{otherwise,} \end{cases} \quad (5.3)$$

is a valid subgradient of  $\mathcal{C}$  at  $x^t$ . On the other hand,  $y - Ax^t = z^t - \omega_t z^{t-1}$ . We finally get

$$\begin{aligned} \text{sg}(\mathcal{C}, x^t) &= \frac{1}{\theta_{t-1}} [\lambda \theta_{t-1} s^t - \theta_{t-1} A^*(z^t - \omega_t z^{t-1})] \\ &= \frac{1}{\theta_{t-1}} [\lambda \theta_{t-1} s^t - \theta_{t-1} (1 - \omega_t) A^* z^{t-1}] - A^*(z^t - z^{t-1}) \\ &= \underbrace{\frac{1}{\theta_{t-1}} [\lambda \theta_{t-1} s^t - \lambda A^* z^{t-1}]}_{(I)} - A^*(z^t - z^{t-1}) + \frac{[\lambda - \theta_{t-1}(1 - \omega_t)]}{\theta_{t-1}} A^* z^{t-1}. \end{aligned}$$

It is straightforward to see from Eqs. (5.2) and (5.3) that  $(I) = \lambda(x^{t-1} - x^t)$ . Hence,

$$\frac{1}{\sqrt{N}} \|\text{sg}(\mathcal{C}, x^t)\| \leq \frac{\lambda}{\theta_{t-1} \sqrt{N}} \|x^t - x^{t-1}\| + \frac{\sigma_{\max}(A)}{\sqrt{N}} \|z^t - z^{t-1}\| + \frac{|\lambda - \theta_{t-1}(1 - \omega_t)|}{\theta_{t-1}} \frac{1}{\sqrt{N}} \|z^{t-1}\|.$$

By Lemma 4.3, and the fact that  $\sigma_{\max}(A)$  is almost surely bounded as  $N \rightarrow \infty$  (cf. Theorem F.2), we deduce that the two terms  $\lambda \|x^t - x^{t-1}\|/(\theta_{t-1} \sqrt{N})$  and  $\sigma_{\max}(A) \|z^t - z^{t-1}\|/\sqrt{N}$  converge to 0 when  $N \rightarrow \infty$  and then  $t \rightarrow \infty$ . For the third term, using state evolution (see Lemma 4.1), we obtain  $\lim_{N \rightarrow \infty} \|z^{t-1}\|^2/N < \infty$ . Finally, using the calibration relation Eq. (1.11), we get

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \left| \frac{\lambda - \theta_{t-1}(1 - \omega_t)}{\theta_{t-1}} \right| \stackrel{\text{a.s.}}{=} \frac{1}{\theta_*} \left| \lambda - \theta_*(1 - \frac{1}{\delta} \mathbb{E} \{ \eta'(X_0 + \tau_* Z; \theta_*) \}) \right| = 0,$$

which finishes the proof.  $\square$

## 5.3 Proof of Lemma 3.5

The proof uses the representation (4.9), together with the expression (4.10) for the conditional expectation. Apart from the matrices  $Y_t$ ,  $Q_t$ ,  $X_t$ ,  $M_t$  introduced there, we will also use

$$B_t \equiv [b^0 | b^1 | \dots | b^{t-1}], \quad H_t \equiv [h^1 | h^2 | \dots | h^t].$$

In this section, since  $t$  is fixed, we will drop everywhere the subscript  $t$  from such matrices.

We state below a somewhat more convenient description.

**Lemma 5.1.** *For any  $v \in \mathbb{R}^N$ , we have*

$$Av|_{\mathfrak{S}} = Y(Q^*Q)^{-1}Q^*P_Q v + M(M^*M)^{-1}X^*P_Q^\perp v + P_M^\perp \tilde{A}P_Q^\perp v. \quad (5.4)$$

*Proof.* It is clearly sufficient to prove that, for  $v = v_\parallel + v_\perp$ ,  $P_Q v_\parallel = v_\parallel$ ,  $P_Q^\perp v_\perp = v_\perp$ , we have

$$Ev_\parallel = Y(Q^*Q)^{-1}Q^*v_\parallel, \quad Ev_\perp = M(M^*M)^{-1}X^*v_\perp. \quad (5.5)$$

The first identity is an easy consequence of the fact that  $X^*Q = M^*AQ = M^*Y$ , while the second one follows immediately from  $Q^*v_\perp = 0$ .  $\square$

The following fact (see Appendix D for a proof) will be used several times.

**Lemma 5.2.** *For any  $t$  there exists  $c > 0$  such that, for  $R \in \{Q^*Q; M^*M; X^*X; Y^*Y\}$ , as  $N \rightarrow \infty$  almost surely,*

$$c \leq \lambda_{\min}(R/N) \leq \lambda_{\max}(R/N) \leq 1/c. \quad (5.6)$$

Given the above remarks, we will immediately see that Lemma 3.5 is implied by the following statement.

**Lemma 5.3.** *Let  $S \subseteq [N]$  be given such that  $|S| \leq N(\delta - \gamma)$ , for some  $\gamma > 0$ . Then there exists  $\alpha_1 = \alpha_1(\gamma) > 0$  (independent of  $t$ ) and  $\alpha_2 = \alpha_2(\gamma, t) > 0$  (depending on  $t$  and  $\gamma$ ) such that*

$$\mathbb{P}\left\{\min_{\|v\|=1, \text{supp}(v) \subseteq S} \|Ev + P_M^\perp \tilde{A}P_Q^\perp v\| \leq \alpha_2 \mid \mathfrak{S}_t\right\} \leq e^{-N\alpha_1},$$

*with probability (over  $\mathfrak{S}_t$ ) converging to 1 as  $t \rightarrow \infty$ . (With  $Ev = Y(Q^*Q)^{-1}Q^*P_Q v + M(M^*M)^{-1}X^*P_Q^\perp v$ .)*

In the next section we will show that this lemma implies Lemma 3.5. We will then prove the lemma just stated.

### 5.3.1 Lemma 5.3 implies Lemma 3.5

We need to show that, for  $S$  measurable on  $\mathfrak{S}_t$  and  $|S| \leq N(\delta - c)$  there exist  $a_1 = a_1(c) > 0$  and  $a_2 = a_2(c, t) > 0$  such that

$$\lim_{N \rightarrow \infty} \mathbb{P}\left\{\min_{|S'| \leq a_1 N} \min_{\|v\|=1, \text{supp}(v) \subseteq S \cup S'} \|Av\| < a_2\right\} = 0.$$

Conditioning on  $\mathfrak{S}_t$  and using the union bound, this probability can be estimated as

$$\begin{aligned} \mathbb{E}\left\{\mathbb{P}\left\{\min_{|S'| \leq a_1 N} \min_{\|v\|=1, \text{supp}(v) \subseteq S \cup S'} \|Av\| < a_2 \mid \mathfrak{S}_t\right\}\right\} &\leq \\ &\leq e^{Nh(a_1)} \mathbb{E}\left\{\max_{|S'| \leq a_1 N} \mathbb{P}\left\{\min_{\|v\|=1, \text{supp}(v) \subseteq S \cup S'} \|Av\| < a_2 \mid \mathfrak{S}_t\right\}\right\}, \end{aligned}$$

where  $h(p) = -p \log p - (1-p) \log(1-p)$  is the binary entropy function. The union bound calculation indeed proceeds as follows

$$\begin{aligned} \mathbb{P}\left\{\min_{|S'|\leq Na_1} \mathbf{X}_{S'} < a_2 \middle| \mathfrak{S}_t\right\} &\leq \sum_{|S'|\leq Na_1} \mathbb{P}\{\mathbf{X}_{S'} < a_2 \middle| \mathfrak{S}_t\} \\ &\leq \left[\sum_{k=1}^{Na_1} \binom{N}{k}\right] \max_{|S'|\leq Na_1} \mathbb{P}\{\mathbf{X}_{S'} < a_2 \middle| \mathfrak{S}_t\} \\ &\leq e^{Nh(a_1)} \max_{|S'|\leq Na_1} \mathbb{P}\{\mathbf{X}_{S'} < a_2 \middle| \mathfrak{S}_t\}, \end{aligned}$$

where  $\mathbf{X}_{S'} = \min_{\|v\|=1, \text{supp}(v) \subseteq S \cup S'} \|Av\|$ . Now, fix  $a_1 < c/2$  in such a way that  $h(a_1) \leq \alpha_1(c/2)/2$  (with  $\alpha_1$  defined as per Lemma 5.3). Further choose  $a_2 = \alpha_2(c/2, t)/2$ . The above probability is then upper bounded by

$$e^{N\alpha_1(c/2)/2} \mathbb{E}\left\{\max_{|S''|\leq N(\delta-c/2)} \mathbb{P}\left\{\min_{\|v\|=1, \text{supp}(v) \subseteq S''} \|Av\| < \frac{1}{2}\alpha_2(c/2, t) \middle| \mathfrak{S}_t\right\}\right\}.$$

Finally, applying Lemma 5.3 and using Lemma 5.1 to estimate  $Av$ , we get

$$e^{N\alpha_1/2} \mathbb{E}\left\{\max_{|S''|\leq N(\delta-c/2)} e^{-N\alpha_1}\right\} \rightarrow 0.$$

This finishes the proof.  $\square$

### 5.3.2 Proof of Lemma 5.3

We begin with the following Pythagorean inequality.

**Lemma 5.4.** *Let  $S \subseteq [N]$  be given such that  $|S| \leq N(\delta - \gamma)$ , for some  $\gamma > 0$ . Recall that  $Ev = Y(Q^*Q)^{-1}Q^*P_Q v + M(M^*M)^{-1}X^*P_Q^\perp v$  and consider the event*

$$\mathcal{E}_1 \equiv \left\{\|Ev + P_M^\perp \tilde{A}P_Q^\perp v\|^2 \geq \frac{\gamma}{4\delta} \|Ev - P_M \tilde{A}P_Q^\perp v\|^2 + \frac{\gamma}{4\delta} \|\tilde{A}P_Q^\perp v\|^2 \quad \forall v \text{ s.t. } \|v\| = 1 \text{ and } \text{supp}(v) \subseteq S\right\}.$$

*Then there exists  $a = a(\gamma) > 0$  such that  $\mathbb{P}\{\mathcal{E}_1 | \mathfrak{S}_t\} \geq 1 - e^{-Na}$ .*

*Proof.* We claim that the following inequality holds for all  $v \in \mathbb{R}^N$ , that satisfy  $\|v\| = 1$  and  $\text{supp}(v) \subseteq S$ , with the probability claimed in the statement

$$|(Ev - P_M \tilde{A}P_Q^\perp v, \tilde{A}P_Q^\perp v)| \leq \sqrt{1 - \frac{\gamma}{2\delta}} \|Ev - P_M \tilde{A}P_Q^\perp v\| \|\tilde{A}P_Q^\perp v\|. \quad (5.7)$$

Here the notation  $(u, v)$  refers to the usual scalar product  $u^*v$  of vectors  $u$  and  $v$  of the same dimension. Assuming that the claim holds, we have indeed

$$\begin{aligned} \|Ev + P_M^\perp \tilde{A}P_Q^\perp v\|^2 &\geq \|Ev - P_M \tilde{A}P_Q^\perp v\|^2 + \|\tilde{A}P_Q^\perp v\|^2 - 2|(Ev - P_M \tilde{A}P_Q^\perp v, \tilde{A}P_Q^\perp v)| \\ &\geq \|Ev\|^2 + \|P_M^\perp \tilde{A}P_Q^\perp v\|^2 - 2\sqrt{1 - \frac{\gamma}{2\delta}} \|Ev - P_M \tilde{A}P_Q^\perp v\| \|\tilde{A}P_Q^\perp v\| \\ &\geq \left(1 - \sqrt{1 - \frac{\gamma}{2\delta}}\right) \left\{\|Ev - P_M \tilde{A}P_Q^\perp v\|^2 + \|\tilde{A}P_Q^\perp v\|^2\right\}, \end{aligned}$$

which implies the thesis.

In order to prove the claim (5.7), we notice that for any  $v$ , the unit vector  $\tilde{A}P_Q^\perp v / \|\tilde{A}P_Q^\perp v\|$  belongs to the random linear space  $\text{im}(\tilde{A}P_Q^\perp P_S)$ . Here  $P_S$  is the orthogonal projector onto the subspace of vectors supported on  $S$ . Further  $\text{im}(\tilde{A}P_Q^\perp P_S)$  is a uniformly random subspace of dimension at most  $N(\delta - \gamma)$ . Also, the normalized vector  $(Ev - P_M \tilde{A}P_Q^\perp v) / \|Ev - P_M \tilde{A}P_Q^\perp v\|$  belongs to the linear space of dimension at most  $2t$  spanned the columns of  $M$  and of  $B$ . The claim follows then from a standard concentration-of-measure argument. In particular applying Proposition E.1 for

$$m = n, \quad m\lambda = N(\delta - \gamma), \quad d = 2t \quad \text{and} \quad \varepsilon = \sqrt{1 - \frac{\gamma}{2\delta}} - \sqrt{1 - \frac{\gamma}{\delta}}$$

yields

$$\left( \frac{Ev - P_M \tilde{A}P_Q^\perp v}{\|Ev - P_M \tilde{A}P_Q^\perp v\|}, \frac{\tilde{A}P_Q^\perp v}{\|\tilde{A}P_Q^\perp v\|} \right) \leq \sqrt{\lambda} + \varepsilon = \sqrt{1 - \frac{\gamma}{2\delta}}.$$

(Notice that in Proposition E.1 is stated for the equivalent case of a random sub-space of fixed dimension  $d$ , and a subspace of dimension scaling linearly with the ambient one.)  $\square$

Next we estimate the term  $\|\tilde{A}P_Q^\perp v\|^2$  in the above lower bound.

**Lemma 5.5.** *Let  $S \subseteq [N]$  be given such that  $|S| \leq N(\delta - \gamma)$ , for some  $\gamma > 0$ . Then there exists constant  $c_1 = c_1(\gamma)$ ,  $c_2 = c_2(\gamma)$  such that the event*

$$\mathcal{E}_2 \equiv \left\{ \|\tilde{A}P_Q^\perp v\| \geq c_1(\gamma) \|P_Q^\perp v\| \quad \forall v \text{ such that } \text{supp}(v) \subseteq S \right\},$$

holds with probability  $\mathbb{P}\{\mathcal{E}_2 | \mathfrak{S}_t\} \geq 1 - e^{-Nc_2}$ .

*Proof.* Let  $V$  be the linear space  $V = \text{im}(P_Q^\perp P_S)$ . Of course the dimension of  $V$  is at most  $N(\delta - \gamma)$ . Then we have (for all vectors with  $\text{supp}(v) \subseteq S$ )

$$\|\tilde{A}P_Q^\perp v\| \geq \sigma_{\min}(\tilde{A}|_V) \|P_Q^\perp v\|, \quad (5.8)$$

where  $\tilde{A}|_V$  is the restriction of  $\tilde{A}$  to the subspace  $V$ . By invariance of the distribution of  $\tilde{A}$  under rotation,  $\sigma_{\min}(\tilde{A}|_V)$  is distributed as the minimum singular value of a gaussian matrix of dimensions  $N\delta \times \dim(V)$ . The latter is almost surely bounded away from 0 as  $N \rightarrow \infty$ , since  $\dim(V) \leq N(\delta - \gamma)$  (see for instance Theorem F.2). Large deviation estimates [LPRTJ05] imply that the probability that the minimum singular value is smaller than a constant  $c_1(\gamma)$  is exponentially small.  $\square$

Finally a simple bound to control the norm of  $Ev$ .

**Lemma 5.6.** *There exists a constant  $c = c(t) > 0$  such that, defining the event,*

$$\mathcal{E}_3 \equiv \left\{ \|EP_Q v\| \geq c(t) \|P_Q v\|, \|EP_Q^\perp v\| \leq c(t)^{-1} \|P_Q^\perp v\|, \text{ for all } v \in \mathbb{R}^N \right\}, \quad (5.9)$$

we have  $\mathbb{P}(\mathcal{E}_3) \rightarrow 1$  as  $N \rightarrow \infty$ .

*Proof.* Without loss of generality take  $v = Qa$  for  $a \in \mathbb{R}^t$ . By Lemma 5.1 we have  $\|EP_Q v\|^2 = \|Ya\|^2 \geq \lambda_{\min}(Y^*Y) \|a\|^2$ . Analogously  $\|P_Q v\|^2 = \|Qa\|^2 \leq \lambda_{\max}(Q^*Q) \|a\|^2$ . The bound  $\|EP_Q v\| \geq c(t) \|P_Q v\|$  follows then from Lemma 5.2.

The bound  $\|EP_Q^\perp v\| \leq c(t)^{-1} \|P_Q^\perp v\|$  is proved analogously.  $\square$

We can now prove Lemma 5.3 as promised.

*Proof of Lemma 5.3.* By Lemma 5.6 we can assume that event  $\mathcal{E}_3$  holds, for some function  $c = c(t)$  (without loss of generality  $c < 1/2$ ). We will let  $\mathcal{E}$  be the event

$$\mathcal{E} \equiv \left\{ \min_{\|v\|=1, \text{supp}(v) \subseteq S} \|Ev + P_M^\perp \tilde{A} P_Q^\perp v\| \leq \alpha_2(t) \right\}. \quad (5.10)$$

for  $\alpha_2(t) > 0$  small enough.

Let us assume first that  $\|P_Q^\perp v\| \leq c^2/10$ , whence

$$\begin{aligned} \|Ev - P_M \tilde{A} P_Q^\perp v\| &\geq \|EP_Q v\| - \|EP_Q^\perp v\| - \|P_M \tilde{A} P_Q^\perp v\| \\ &\geq c\|P_Q v\| - (c^{-1} + \|\tilde{A}\|_2)\|P_Q^\perp v\| \\ &\geq \frac{c}{2} - \frac{c}{10} - \|\tilde{A}\|_2 \frac{c^2}{10} = \frac{2c}{5} - \|\tilde{A}\|_2 \frac{c^2}{10}, \end{aligned}$$

where the last inequality uses  $\|P_Q v\| = \sqrt{1 - \|P_Q^\perp v\|^2} \geq 1/2$ . Therefore, using Lemma 5.4, we get

$$\mathbb{P}\{\mathcal{E}|\mathfrak{S}_t\} \leq \mathbb{P}\left\{\frac{2c}{5} - \|\tilde{A}\|_2 \frac{c^2}{10} \leq \sqrt{\frac{4\delta}{\gamma}} \alpha_2(t) \middle| \mathfrak{S}_t\right\} + e^{-Na},$$

and the thesis follows from large deviation bounds on the norm  $\|\tilde{A}\|_2$  [Led01] by first taking  $c$  small enough, and then choosing  $\alpha_2(t) < \frac{c}{5} \sqrt{\frac{\gamma}{4\delta}}$ .

Next we assume  $\|P_Q^\perp v\| \geq c^2/10$ . Due to Lemma 5.4 and 5.5 we can assume that events  $\mathcal{E}_1$  and  $\mathcal{E}_2$  hold. Therefore

$$\|Ev + P_M^\perp \tilde{A} P_Q^\perp v\| \geq \left(\frac{\gamma}{4\delta}\right)^{1/2} \|\tilde{A} P_Q^\perp v\| \geq \left(\frac{\gamma}{4\delta}\right)^{1/2} c_1(\gamma) \|P_Q^\perp v\|,$$

which proves our thesis.  $\square$

## 5.4 Proof of Lemma 3.6

The key step consists in establishing the following result, which will be instrumental in the proof of Lemma 4.3 as well (and whose proof is deferred to Appendix C.1).

**Lemma 5.7.** *Assume  $\alpha > \alpha_{\min}(\delta)$  and let  $\{\mathbf{R}_{s,t}\}$  be defined by the recursion (4.13) with initial condition (4.14). Then there exists constants  $\mathbf{B}_1, r_1 > 0$  such that for all  $t \geq 0$*

$$|\mathbf{R}_{t,t} - \tau_*^2| \leq \mathbf{B}_1 e^{-r_1 t}, \quad (5.11)$$

$$|\mathbf{R}_{t,t+1} - \tau_*^2| \leq \mathbf{B}_1 e^{-r_1 t}. \quad (5.12)$$

It is also useful to prove the following fact.

**Lemma 5.8.** *For any  $\alpha > 0$  and  $T \geq 0$ , the  $T \times T$  matrix  $R_{T+1} \equiv \{\mathbf{R}_{s,t}\}_{0 \leq s,t < T}$  is strictly positive definite.*



*Proof.* In proof of Theorem 4.2 we show that

$$R_{s,t} = \lim_{N \rightarrow \infty} \langle h^{s+1}, h^{t+1} \rangle = \lim_{N \rightarrow \infty} \langle m^s, m^t \rangle,$$

almost surely. Hence,  $R_{T+1} \stackrel{\text{a.s.}}{=} \delta \lim_{N \rightarrow \infty} (M_{T+1}^* M_{T+1} / N)$ . Thus the result follows from Lemma 5.2.  $\square$

It is then relatively easy to deduce the following.

**Lemma 5.9.** *Assume  $\alpha > \alpha_{\min}(\delta)$  and let  $\{R_{s,t}\}$  be defined by the recursion (4.13) with initial condition (4.14). Then there exists constants  $B_2, r_2 > 0$  such that for all  $t_1, t_2 \geq t \geq 0$*

$$|R_{t_1, t_2} - \tau_*^2| \leq B_2 e^{-r_2 t}. \quad (5.13)$$

*Proof.* By triangular inequality and Eq. (5.11), we have

$$|R_{t_1, t_2} - \tau_*^2| \leq \frac{1}{2} |R_{t_1, t_1} - 2R_{t_1, t_2} + R_{t_2, t_2}| + B_1 e^{-r_1 t}. \quad (5.14)$$

By Lemma 5.8 there exist gaussian random variables  $Z_0, Z_1, Z_2, \dots$  on the same probability space with  $\mathbb{E}\{Z_t\} = 0$  and  $\mathbb{E}\{Z_t Z_s\} = R_{t,s}$  (in fact in proof of Theorem 4.2 we show that  $\{Z_i\}_{T \geq i \geq 0}$  is the weak limit of the empirical distribution of  $\{h^{i+1}\}_{T \geq i \geq 0}$ ). Then (assuming, without loss of generality,  $t_2 > t_1$ ) we have

$$\begin{aligned} |R_{t_1, t_1} - 2R_{t_1, t_2} + R_{t_2, t_2}| &= \mathbb{E}\{(Z_{t_1} - Z_{t_2})^2\} \\ &= \sum_{i,j=t_1}^{t_2-1} \mathbb{E}\{(Z_{i+1} - Z_i)(Z_{j+1} - Z_j)\} \\ &\leq \left[ \sum_{i=t_1}^{t_2-1} \mathbb{E}\{(Z_{i+1} - Z_i)^2\}^{1/2} \right]^2 \\ &\leq 4B_1 \left[ \sum_{i=t_1}^{\infty} e^{-r_1 i/2} \right]^2 \\ &\leq \frac{4B_1}{(1 - e^{-r_1/2})^2} e^{-r_1 t_1}, \end{aligned}$$

which, together with Eq. (5.14) proves our claim.  $\square$

We are now in position to prove Lemma 3.6.

*Proof of Lemma 3.6.* We will show that, under the assumptions of the Lemma,  $\lim_{N \rightarrow \infty} |S_{t_2}(\gamma) \setminus S_{t_1}(\gamma)|/N \leq \xi$  almost surely, which implies our claim. Indeed, by Theorem 4.2 we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} |S_{t_2}(\gamma) \setminus S_{t_1}(\gamma)| &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\{|v_i^{t_2}| \geq 1-\gamma, |v_i^{t_1}| < 1-\gamma\}} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\{|x^{t_2-1} + A^* z^{t_2-1} - x^{t_2}| \geq (1-\gamma)\theta_{t_2-1}, |x^{t_1-1} + A^* z^{t_1-1} - x^{t_1}| < (1-\gamma)\theta_{t_2-1}\}} \\ &= \mathbb{P}\{|X_0 + Z_{t_2-1}| \geq (1-\gamma)\theta_{t_2-1}, |X_0 + Z_{t_1-1}| < (1-\gamma)\theta_{t_1-1}\} \equiv P_{t_1, t_2}, \end{aligned}$$

where  $(Z_{t_1}, Z_{t_2})$  are jointly normal with  $\mathbb{E}\{Z_{t_1}^2\} = R_{t_1, t_1}$ ,  $\mathbb{E}\{Z_{t_1} Z_{t_2}\} = R_{t_1, t_2}$ ,  $\mathbb{E}\{Z_{t_2}^2\} = R_{t_2, t_2}$ . (Notice that, although the function  $\mathbb{I}\{\cdots\}$  is discontinuous, the random vector  $(X_0 + Z_{t_1-1}, X_0 + Z_{t_2-1})$  admits a density and hence Theorem 4.2 applies by weak convergence of the empirical distribution of  $\{(x_i^{t_1-1} + (A^* z^{t_1-1})_i, x_i^{t_2-1} + (A^* z^{t_2-1})_i)\}_{1 \leq i \leq N}$ .)

Let  $a \equiv (1 - \gamma)\alpha\tau_*$ . By Proposition 1.2, for any  $\varepsilon > 0$  and all  $t_*$  large enough we have  $|(1 - \gamma)\theta_{t_i-1} - a| \leq \varepsilon$  for  $i \in \{1, 2\}$ . Then

$$\begin{aligned} P_{t_1, t_2} &\leq \mathbb{P}\{|X_0 + Z_{t_2-1}| \geq a - \varepsilon, |X_0 + Z_{t_1-1}| < a + \varepsilon\} \\ &\leq \mathbb{P}\{|Z_{t_1-1} - Z_{t_2-1}| \geq 2\varepsilon\} + \mathbb{P}\{a - 3\varepsilon \leq |X_0 + Z_{t_1-1}| \leq a + \varepsilon\} \\ &\leq \frac{1}{4\varepsilon^2} [R_{t_1-1, t_1-1} - 2R_{t_1-1, t_2-1} + R_{t_2-1, t_2-1}] + \frac{4\varepsilon}{\sqrt{2\pi R_{t_1-1, t_1-1}}} \\ &\leq \frac{1}{\varepsilon^2} B_2 e^{-r_2 t_*} + \frac{\varepsilon}{\tau_*}, \end{aligned}$$

where the last inequality follows by Lemma 5.9. By taking  $\varepsilon = e^{-r_2 t_*/3}$  we finally get (for some constant  $C$ )  $P_{t_1, t_2} \leq C e^{-r_2 t_*}$ , which implies our claim.  $\square$

## Acknowledgement

It is a pleasure to thank David Donoho and Arian Maleki for many stimulating exchanges. This work was partially supported by a Terman fellowship, the NSF CAREER award CCF-0743978 and the NSF grant DMS-0806211.

## A Properties of the state evolution recursion

### A.1 Proof of Proposition 1.2

It is a straightforward calculus exercise to compute the partial derivatives

$$\frac{\partial F}{\partial \tau^2}(\tau^2, \theta) = \frac{1}{\delta} \mathbb{E}\left\{\Phi\left(\frac{X_0 - \theta}{\tau}\right) + \Phi\left(\frac{-X_0 - \theta}{\tau}\right)\right\} - \frac{1}{\delta} \mathbb{E}\left\{\frac{X_0}{\tau} \phi\left(\frac{X_0 - \theta}{\tau}\right) - \frac{X_0}{\tau} \phi\left(\frac{-X_0 - \theta}{\tau}\right)\right\}, \quad (\text{A.1})$$

$$\frac{\partial F}{\partial \theta}(\tau^2, \theta) = \frac{2\theta}{\delta} \mathbb{E}\left\{\Phi\left(\frac{X_0 - \theta}{\tau}\right) + \Phi\left(\frac{-X_0 - \theta}{\tau}\right)\right\} - \frac{2\tau}{\delta} \mathbb{E}\left\{\phi\left(\frac{X_0 - \theta}{\tau}\right) + \phi\left(\frac{-X_0 - \theta}{\tau}\right)\right\}. \quad (\text{A.2})$$

From these formulae we obtain the total derivative

$$\begin{aligned} \delta \frac{dF}{d\tau^2}(\tau^2, \alpha\tau) &= (1 + \alpha^2) \mathbb{E}\left\{\Phi\left(\frac{X_0 - \alpha\tau}{\tau}\right) + \Phi\left(\frac{-X_0 - \alpha\tau}{\tau}\right)\right\} \\ &\quad - \mathbb{E}\left\{\left(\frac{X_0 + \alpha\tau}{\tau}\right) \phi\left(\frac{X_0 - \alpha\tau}{\tau}\right) - \left(\frac{X_0 - \alpha\tau}{\tau}\right) \phi\left(\frac{-X_0 - \alpha\tau}{\tau}\right)\right\}. \end{aligned} \quad (\text{A.3})$$

Differentiating once more

$$\delta \frac{d^2 F}{d(\tau^2)^2}(\tau^2, \alpha\tau) = -\frac{1}{2\tau^2} \mathbb{E}\left\{\left(\frac{X_0}{\tau}\right)^3 \left[\phi\left(\frac{X_0 - \alpha\tau}{\tau}\right) - \phi\left(\frac{-X_0 - \alpha\tau}{\tau}\right)\right]\right\}.$$

Now we have

$$u^3[\phi(u - \alpha) - \phi(-u - \alpha)] \geq 0, \quad (\text{A.4})$$

with the inequality being strict whenever  $\alpha > 0$ ,  $u \neq 0$ . It follows that  $\tau^2 \mapsto F(\tau^2, \alpha\tau)$  is concave, and strictly concave provided  $\alpha > 0$  and  $X_0$  is not identically 0.

From Eq. (A.3) we obtain

$$\lim_{\tau^2 \rightarrow \infty} \frac{dF}{d\tau^2}(\tau^2, \alpha\tau) = \frac{2}{\delta} \{ (1 + \alpha^2)\Phi(-\alpha) - \alpha\phi(\alpha) \}, \quad (\text{A.5})$$

which is strictly positive for all  $\alpha \geq 0$ . To see this, let  $f(\alpha) \equiv (1 + \alpha^2)\Phi(-\alpha) - \alpha\phi(\alpha)$ , and notice that  $f'(\alpha) = 2\alpha\Phi(-\alpha) - 2\phi(\alpha) < 0$ , and  $f(\infty) = 0$ .

Since  $\tau^2 \mapsto F(\tau^2, \alpha\tau)$  is concave, and strictly increasing for  $\tau^2$  large enough, it also follows that it is increasing everywhere.

Notice that  $\alpha \mapsto f(\alpha)$  is strictly decreasing with  $f(0) = 1/2$ . Hence, for  $\alpha > \alpha_{\min}(\delta)$ , we have  $F(\tau^2, \alpha\tau) > \tau^2$  for  $\tau^2$  small enough and  $F(\tau^2, \alpha\tau) < \tau^2$  for  $\tau^2$  large enough. Therefore the fixed point equation admits at least one solution. It follows from the concavity of  $\tau^2 \mapsto F(\tau^2, \alpha\tau)$  that the solution is unique and that the sequence of iterates  $\tau_t^2$  converge to  $\tau_*$ .  $\square$

## A.2 Proof of Proposition 1.3

As a first step, we claim that  $\alpha \mapsto \tau_*^2(\alpha)$  is continuously differentiable on  $(0, \infty)$ . Indeed this is defined as the unique solution of

$$\tau_*^2 = F(\tau_*^2, \alpha\tau_*). \quad (\text{A.6})$$

Since  $(\tau^2, \alpha) \mapsto F(\tau^2, \alpha\tau)$  is continuously differentiable and  $0 \leq \frac{dF}{d\tau^2}(\tau_*^2, \alpha\tau_*) < 1$  (the second inequality being a consequence of concavity plus  $\lim_{\tau^2 \rightarrow \infty} \frac{dF}{d\tau^2}(\tau^2, \alpha\tau) < 1$ , both shown in the proof of Proposition 1.2), the claim follows from the implicit function theorem applied to the mapping  $(\tau^2, \alpha) \mapsto [\tau^2 - F(\tau^2, \alpha)]$ .

Next notice that  $\tau_*^2(\alpha) \rightarrow +\infty$  as  $\alpha \downarrow \alpha_{\min}(\delta)$ . Indeed, introducing the notation  $F'_\infty \equiv \lim_{\tau^2 \rightarrow \infty} \frac{dF}{d\tau^2}(\tau^2, \alpha\tau)$ , we have, again by concavity,

$$\tau_*^2 \geq F(0, 0) + F'_\infty \tau_*^2,$$

i.e.  $\tau_*^2 \geq F(0, 0)/(1 - F'_\infty)$ . Now  $F(0, 0) \geq \sigma^2$ , while  $F'_\infty \uparrow 1$  as  $\alpha \downarrow \alpha_{\min}(\delta)$  (shown in the proof of Proposition 1.2), whence the claim follows.

Finally  $\tau_*^2(\alpha) \rightarrow \sigma^2 + \mathbb{E}\{X_0^2\}/\delta$  as  $\alpha \rightarrow \infty$ . Indeed for any fixed  $\tau^2 > 0$  we have  $F(\tau^2, \alpha\tau) \rightarrow \sigma^2 + \mathbb{E}\{X_0^2\}/\delta$  as  $\alpha \rightarrow \infty$  whence the claim follows by uniqueness of  $\tau_*$ .

Next consider the function  $(\alpha, \tau^2) \mapsto g(\alpha, \tau^2)$  defined by

$$g(\alpha, \tau^2) \equiv \alpha\tau \left\{ 1 - \frac{1}{\delta} \mathbb{P}\{|X_0 + \tau Z| \geq \alpha\tau\} \right\}.$$

Notice that  $\lambda(\alpha) = g(\alpha, \tau_*^2(\alpha))$ . Since  $g$  is continuously differentiable, it follows that  $\alpha \mapsto \lambda(\alpha)$  is continuously differentiable as well.

Next consider  $\alpha \downarrow \alpha_{\min}$ , and let  $l(\alpha) \equiv 1 - \frac{1}{\delta} \mathbb{P}\{|X_0 + \tau_* Z| \geq \alpha \tau_*\}$ . Since  $\tau_* \rightarrow +\infty$  in this limit, we have

$$l_* \equiv \lim_{\alpha \rightarrow \alpha_{\min}^+} l(\alpha) = 1 - \frac{1}{\delta} \mathbb{P}\{|Z| \geq \alpha_{\min}\} = 1 - \frac{2}{\delta} \Phi(-\alpha_{\min}).$$

Using the characterization of  $\alpha_{\min}$  in Eq. (1.10) (and the well known inequality  $\alpha \Phi(-\alpha) \leq \phi(\alpha)$  valid for all  $\alpha > 0$ ), it is immediate to show that  $l_* < 0$ . Therefore

$$\lim_{\alpha \rightarrow \alpha_{\min}^+} \lambda(\alpha) = l_* \lim_{\alpha \rightarrow \alpha_{\min}^+} \alpha \tau_*(\alpha) = -\infty.$$

Finally let us consider the limit  $\alpha \rightarrow \infty$ . Since  $\tau_*(\alpha)$  remains bounded, we have  $\lim_{\alpha \rightarrow \infty} \mathbb{P}\{|X_0 + \tau_* Z| \geq \alpha \tau_*\} = 0$  whence

$$\lim_{\alpha \rightarrow \infty} \lambda(\alpha) = \lim_{\alpha \rightarrow \infty} \alpha \tau_*(\alpha) = \infty.$$

□

### A.3 Proof of Corollary 1.5

By Proposition 1.3, it is sufficient to prove that, for any  $\lambda > 0$  there exists a unique  $\alpha > \alpha_{\min}$  such that  $\lambda(\alpha) = \lambda$ . Assume by contradiction that there are two distinct such values  $\alpha_1, \alpha_2$ .

Notice that in this case, the function  $\alpha(\lambda)$  is not defined uniquely and we can apply Theorem 1.4 to both choices  $\alpha(\lambda) = \alpha_1$  and  $\alpha(\lambda) = \alpha_2$ . Using the test function  $\psi(x, y) = (x - y)^2$  we deduce that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \|\hat{x} - x_0\|^2 = \mathbb{E}\{\left[\eta(X_0 + \tau_* Z; \alpha \tau_*) - X_0\right]^2\} = \delta(\tau_*^2 - \sigma^2).$$

Since the left hand side does not depend on the choice of  $\alpha$ , it follows that  $\tau_*(\alpha_1) = \tau_*(\alpha_2)$ .

Next apply Theorem 1.4 to the function  $\psi(x, y) = |x|$ . We get

$$\lim_{N \rightarrow \infty} \frac{1}{N} \|\hat{x}\|_1 = \mathbb{E}\{|\eta(X_0 + \tau_* Z; \alpha \tau_*)|\}.$$

For fixed  $\tau_*$ ,  $\theta \mapsto \mathbb{E}\{|\eta(X_0 + \tau_* Z; \theta)|\}$  is strictly decreasing in  $\theta$ . It follows that  $\alpha_1 \tau_*(\alpha_1) = \alpha_2 \tau_*(\alpha_2)$ . Since we already proved that  $\tau_*(\alpha_1) = \tau_*(\alpha_2)$ , we conclude  $\alpha_1 = \alpha_2$ . □

## B Proof of Theorem 4.2

First note that using representation (4.2) we have  $x^t + A^* z^t = x_0 - h^{t+1}$ . Furthermore, using Lemma F.3(b) we have almost surely

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_{0,i} - h_i^{s+1}, x_{0,i} - h_i^{t+1}, x_{0,i}) &= \mathbb{E}\left\{\psi(X_0 - \tilde{Z}_s, X_0 - \tilde{Z}_t, X_0)\right\} \\ &= \mathbb{E}\left\{\psi(X_0 + \tilde{Z}_s, X_0 + \tilde{Z}_t, X_0)\right\} \end{aligned}$$

for gaussian variables  $\tilde{Z}_s, \tilde{Z}_t$  that have zero mean and are independent of  $X_0$ . Define for all  $s \geq 0$  and  $t \geq 0$ ,

$$\tilde{R}_{t,s} \equiv \lim_{N \rightarrow \infty} \langle h^{t+1}, h^{s+1} \rangle = \mathbb{E}\{\tilde{Z}_t \tilde{Z}_s\}. \quad (\text{B.1})$$

Therefore, all we need to show is that for all  $s, t \geq 0$ :  $R_{t,s}$  and  $\tilde{R}_{t,s}$  are equal. We prove this by induction on  $\max(s, t)$ .

- For  $s = t = 0$  we have using Lemma F.3(b) almost surely

$$\tilde{R}_{0,0} \equiv \lim_{N \rightarrow \infty} \langle h^1, h^1 \rangle = \tau_0^2 = \sigma^2 + \frac{1}{\delta} \mathbb{E}\{X_0^2\},$$

that is equal to  $R_{0,0}$ .

- *Induction hypothesis:* Assume that for all  $s \leq k$  and  $t \leq k$ ,

$$R_{t,s} = \tilde{R}_{t,s}. \quad (\text{B.2})$$

- Then we prove Eq. (B.2) for  $t = k + 1$  (case  $s = k + 1$  is similar). First assume  $s = 0$  and  $t = k + 1$  in which using Lemma F.3(c) we have almost surely

$$\begin{aligned} \tilde{R}_{k+1,0} &= \lim_{N \rightarrow \infty} \langle h^{k+2}, h^1 \rangle = \lim_{n \rightarrow \infty} \langle m^{k+1}, m^0 \rangle \\ &= \lim_{n \rightarrow \infty} \langle b^{k+1} - w, b^0 - w \rangle = \sigma^2 + \frac{1}{\delta} \lim_{N \rightarrow \infty} \langle q^{k+1}, q^0 \rangle \\ &= \sigma^2 + \frac{1}{\delta} \mathbb{E} \left\{ [\eta(X_0 - \tilde{Z}_k; \theta_k) - X_0] [-X_0] \right\}, \\ &= \sigma^2 + \frac{1}{\delta} \mathbb{E} \left\{ [\eta(X_0 + \tilde{Z}_k; \theta_k) - X_0] [-X_0] \right\}, \end{aligned}$$

where the last equality uses  $q^0 = -x_0$  and Lemma F.3(b) for the pseudo-Lipschitz function  $(h_i^{k+1}, x_{0,i}) \mapsto [\eta(x_{0,i} - h_i^{k+1}; \theta_k) - x_{0,i}] [-x_{0,i}]$ . Here  $X_0 \sim p_{X_0}$  and  $\tilde{Z}_k$  are independent and the latter is mean zero gaussian with  $\mathbb{E}\{\tilde{Z}_k^2\} = \tilde{R}_{k,k}$ . But using the induction hypothesis,  $\tilde{R}_{k,k} = R_{k,k}$  holds. Hence, we can apply Eq. (4.14) to obtain  $\tilde{R}_{t,0} = R_{t,0}$ .

Similarly, for the case  $t = k + 1$  and  $s > 0$ , using Lemma F.3(b)(c) we have almost surely

$$\begin{aligned} \tilde{R}_{k+1,s} &= \lim_{N \rightarrow \infty} \langle h^{k+2}, h^{s+1} \rangle = \lim_{n \rightarrow \infty} \langle m^{k+1}, m^s \rangle \\ &= \lim_{n \rightarrow \infty} \langle b^{k+1} - w, b^s - w \rangle = \sigma^2 + \frac{1}{\delta} \lim_{N \rightarrow \infty} \langle q^{k+1}, q^s \rangle \\ &= \sigma^2 + \frac{1}{\delta} \mathbb{E} \{ [\eta(X_0 + \tilde{Z}_k; \theta_k) - X_0] [\eta(X_0 + \tilde{Z}_{s-1}; \theta_{s-1}) - X_0] \}, \end{aligned}$$

for  $X_0 \sim p_{X_0}$  independent of zero mean gaussian variables  $\tilde{Z}_k$  and  $\tilde{Z}_{s-1}$  that satisfy

$$R_{k,s-1} = \mathbb{E}\{\tilde{Z}_k \tilde{Z}_{s-1}\}, \quad R_{k,k} = \mathbb{E}\{\tilde{Z}_k^2\}, \quad R_{s-1,s-1} = \mathbb{E}\{\tilde{Z}_{s-1}^2\},$$

using the induction hypothesis. Hence the result follows.

## C Proof of Lemma 4.3

The proof of Lemma 4.3 relies on Lemma 5.7 which we will prove in the first subsection.

### C.1 Proof of Lemma 5.7

Before proving Lemma 5.7, we state and prove the following property of gaussian random variables.

**Lemma C.1.** *Let  $Z_1$  and  $Z_2$  be jointly gaussian random variables with  $\mathbb{E}(Z_1^2) = \mathbb{E}(Z_2^2) = 1$  and  $\mathbb{E}(Z_1 Z_2) = c \geq 0$ . Let  $I$  be a measurable subset of the real line. Then  $\mathbb{P}(Z_1 \in I, Z_2 \in I)$  is an increasing function of  $c \in [0, 1]$ .*

*Proof.* Let  $\{X_s\}_{s \in \mathbb{R}}$  be the standard Ornstein-Uhlenbeck process. Then  $(Z_1, Z_2)$  is distributed as  $(X_0, X_t)$  for  $t$  satisfying  $c = e^{-2t}$ . Hence

$$\mathbb{P}(Z_1 \in I, Z_2 \in I) = \mathbb{E}[f(X_0)f(X_t)], \quad (\text{C.1})$$

for  $f$  the indicator function of  $I$ . Since the Ornstein-Uhlenbeck process is reversible with respect to the standard gaussian measure  $\mu_G$ , we have

$$\mathbb{E}[f(X_0)f(X_t)] = \sum_{\ell=0}^{\infty} e^{-\lambda_\ell t} (\psi_\ell, f)_{\mu_G}^2 = \sum_{\ell=0}^{\infty} c^{\frac{\lambda_\ell}{2}} (\psi_\ell, f)_{\mu_G}^2 \quad (\text{C.2})$$

with  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$  the eigenvalues of its generator,  $\{\psi_\ell\}_{\ell \geq 0}$  the corresponding eigenvectors and  $(\cdot, \cdot)_{\mu_G}$  the scalar product in  $L^2(\mu_G)$ . The thesis follows.  $\square$

We now pass to the proof of Lemma 5.7.

*Proof of Lemma 5.7.* It is convenient to change coordinates and define

$$y_{t,1} \equiv R_{t-1,t-1} = \tau_{t-1}^2, \quad y_{t,2} \equiv R_{t,t} = \tau_t^2, \quad y_{t,3} \equiv R_{t-1,t-1} - 2R_{t,t-1} + R_{t,t}. \quad (\text{C.3})$$

The vector  $y_t = (y_{t,1}, y_{t,2}, y_{t,3})$  belongs to  $\mathbb{R}_+^3$  by Lemma 5.8. Using Eq. (4.13), it is immediate to see that this is updated according to the mapping

$$\begin{aligned} y_{t+1} &= G(y_t), \\ G_1(y_t) &\equiv y_{t,2}, \end{aligned} \quad (\text{C.4})$$

$$G_2(y_t) \equiv \sigma^2 + \frac{1}{\delta} \mathbb{E}\{[\eta(X_0 + Z_t; \alpha\sqrt{y_{t,2}}) - X_0]^2\}, \quad (\text{C.5})$$

$$G_3(y_t) \equiv \frac{1}{\delta} \mathbb{E}\{[\eta(X_0 + Z_t; \alpha\sqrt{y_{t,2}}) - \eta(X_0 + Z_{t-1}; \alpha\sqrt{y_{t,1}})]^2\}. \quad (\text{C.6})$$

where  $(Z_t, Z_{t-1})$  are jointly gaussian with zero mean and covariance determined by  $\mathbb{E}\{Z_t^2\} = y_{t,2}$ ,  $\mathbb{E}\{Z_{t-1}^2\} = y_{t,1}$ ,  $\mathbb{E}\{(Z_t - Z_{t-1})^2\} = y_{t,3}$ . This mapping is defined for  $y_{t,3} \leq 2(y_{t,1} + y_{t,2})$ .

Next we will show that by induction on  $t$  that the stronger inequality  $y_{t,3} < (y_{t,1} + y_{t,2})$  holds for all  $t$ . We have indeed

$$y_{t+1,1} + y_{t+1,2} - y_{t+1,3} = 2\sigma^2 + \frac{2}{\delta} \mathbb{E}\{\eta(X_0 + Z_t; \alpha\sqrt{y_{t,2}}) \eta(X_0 + Z_{t-1}; \alpha\sqrt{y_{t,1}})\}.$$

Since  $\mathbb{E}\{Z_t Z_{t-1}\} = (y_{t,1} + y_{t,2} - y_{t,3})/2$  and  $x \mapsto \eta(x; \theta)$  is monotone, we deduce that  $y_{t,3} < (y_{t,1} + y_{t,2})$  implies that  $Z_t, Z_{t-1}$  are positively correlated. Therefore  $\mathbb{E}\{\eta(X_0 + Z_t; \alpha\sqrt{y_{t,2}}) \eta(X_0 + Z_{t-1}; \alpha\sqrt{y_{t,1}})\} \geq 0$ , which in turn yields  $y_{t+1,3} < (y_{t+1,1} + y_{t+1,2})$ .

The initial condition implied by Eq. (4.14) is

$$\begin{aligned} y_{1,1} &= \sigma^2 + \frac{1}{\delta} \mathbb{E}\{X_0^2\}, \\ y_{1,2} &= \sigma^2 + \frac{1}{\delta} \mathbb{E}\{[\eta(X_0 + Z_0; \theta_0) - X_0]^2\}, \\ y_{1,3} &= \frac{1}{\delta} \mathbb{E}\{\eta(X_0 + Z_0; \theta_0)^2\}, \end{aligned}$$

It is easy to check that these satisfy  $y_{1,3} < y_{1,1} + y_{1,2}$ . (This follows from  $\mathbb{E}\{X_0[X_0 - \eta(X_0 + Z_0; \theta_0)]\} > 0$  because  $x_0 \mapsto x_0 - \mathbb{E}_Z \eta(x_0 + Z_0; \theta_0)$  is monotone increasing.) We can hereafter therefore assume  $y_{t,3} < y_{t,1} + y_{t,2}$  for all  $t$ .

We will consider the above iteration for arbitrary initialization  $y_0$  (satisfying  $y_{0,3} < y_{0,1} + y_{0,2}$ ) and will show the following three facts:

**Fact (i).** As  $t \rightarrow \infty$ ,  $y_{t,1}, y_{t,2} \rightarrow \tau_*^2$ . Further the convergence is monotone.

**Fact (ii).** If  $y_{0,1} = y_{0,2} = \tau_*^2$  and  $y_{0,3} \leq 2\tau_*^2$ , then  $y_{t,1} = y_{t,2} = \tau_*^2$  for all  $t$  and  $y_{t,3} \rightarrow 0$ .

**Fact (iii).** The jacobian  $J = J_G(y_*)$  of  $G$  at  $y_* = (\tau_*^2, \tau_*^2, 0)$  has spectral radius  $\sigma(J) < 1$ .

By simple compactness arguments, Facts (i) and (ii) imply  $y_t \rightarrow y_*$  as  $t \rightarrow \infty$ . (Notice that  $y_{t,3}$  remains bounded since  $y_{t,3} \leq (y_{t,1} + y_{t,2})$  and by the convergence of  $y_{t,1}, y_{t,2}$ .) Fact (iii) implies that convergence is exponentially fast.

*Proof of Fact (i).* Notice that  $y_{t,2}$  evolves independently by  $y_{t+1,2} = G_2(y_t) = F(y_{2,t}, \alpha\sqrt{y_{2,t}})$ , with  $F(\cdot, \cdot)$  the state evolution mapping introduced in Eq. (1.6). It follows from Proposition 1.2 that  $y_{t,2} \rightarrow \tau_*^2$  monotonically for any initial condition. Since  $y_{t+1,1} = y_{t,2}$ , the same happens for  $y_{t,1}$ .

*Proof of Fact (ii).* Consider the function  $G_*(x) = G_3(\tau_*^2, \tau_*^2, x)$ . This is defined for  $x \in [0, 4\tau_*^2]$  but since  $y_{t,3} < y_{t,1} + y_{t,2}$  we will only consider  $G_* : [0, 2\tau_*^2] \rightarrow \mathbb{R}_+$ . Obviously  $G_*(0) = 0$ . Further  $G_*$  can be represented as follows in terms of the independent random variables  $Z, W \sim N(0, 1)$ :

$$G_*(x) = \frac{1}{\delta} \mathbb{E}\{[\eta(X_0 + \sqrt{\tau_*^2 - x/4}Z + (\sqrt{x}/2)W; \alpha\tau_*) - \eta(X_0 + \sqrt{\tau_*^2 - x/4}Z - (\sqrt{x}/2)W; \alpha\tau_*)]^2\} \quad (C.7)$$

A straightforward calculation yields

$$G'_*(x) = \frac{1}{\delta} \mathbb{E}\{\eta'(X_0 + Z_t; \alpha\tau_*) \eta'(X_0 + Z_{t-1}; \alpha\tau_*)\} = \frac{1}{\delta} \mathbb{P}\{|X_0 + Z_t| \geq \alpha\tau_*, |X_0 + Z_{t-1}| \geq \alpha\tau_*\},$$

where  $Z_{t-1} = \sqrt{\tau_*^2 - x^2/4}Z + (x/2)W$ ,  $Z_t = \sqrt{\tau_*^2 - x^2/4}Z - (x/2)W$ . In particular, by Lemma C.1,  $x \mapsto G_*(x)$  is strictly increasing (notice that the covariance of  $Z_{t-1}$  and  $Z_t$  is  $\tau_*^2 - (x/2)$  which is decreasing in  $x$ ). Further

$$G'_*(0) = \frac{1}{\delta} \mathbb{E}\{\eta'(X_0 + \tau_* Z; \alpha\tau_*)\}.$$

Hence, since  $\lambda > 0$  using Eq. (1.11) we have  $G'(0) < 1$ . Finally, by Lemma C.1,  $x \mapsto G'(x)$  is decreasing in  $[0, 2\tau_*)$ . It follows that  $y_{t,3} \leq G'(0)^t y_{0,3} \rightarrow 0$  as claimed.

*Proof of Fact (iii).* From the definition of  $G$ , we have the following expression for the Jacobian

$$J_G(y_*) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & F'(\tau_*^2) & 0 \\ a & G'_*(0) & b \end{pmatrix}$$

where with an abuse of notation we let  $F'(\tau_*^2) \equiv \frac{d}{d\tau^2} F(\tau^2, \alpha\tau) \Big|_{\tau^2=\tau_*^2}$ . Computing the eigenvalues of the above matrix, we get

$$\sigma(J) = \max \{ F'(\tau_*^2), G'_*(0) \}.$$

Since  $G'_*(0) < 1$  as proved above, and  $F(\tau_*^2) < 1$  as per Proposition 1.2, the claim follows.  $\square$

## C.2 Lemma 5.7 implies Lemma 4.3

Using representations (4.4) and (4.3) (i.e.,  $b^t = w - z^t$  and  $q^t = x_0 - x^t$ ) and Lemma F.3(c) we obtain,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \|z^{t+1} - z^t\|_2^2 &= \lim_{n \rightarrow \infty} \frac{1}{n} \|b^{t+1} - b^t\|_2^2 \\ &\stackrel{\text{a.s.}}{=} \frac{1}{\delta} \lim_{N \rightarrow \infty} \frac{1}{N} \|q^{t+1} - q^t\|_2^2 \\ &= \frac{1}{\delta} \lim_{N \rightarrow \infty} \frac{1}{N} \|x^{t+1} - x^t\|_2^2, \end{aligned}$$

where the last equality uses  $q^t = x^t - x_0$ . Therefore, it is sufficient to prove the thesis for  $\|x^{t+1} - x^t\|_2$ . By state evolution, Theorem 4.2, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \|x^{t+1} - x^t\|_2^2 &= \mathbb{E} \{ [\eta(X_0 + Z_t; \theta_t) - \eta(X_0 + Z_{t-1}; \theta_{t-1})]^2 \} \\ &\leq 2(\theta_t - \theta_{t-1})^2 + 2 \mathbb{E} \{ (Z_t - Z_{t-1})^2 \} = 2(\theta_t - \theta_{t-1})^2 + 2(R_{t,t} - 2R_{t,t-1} + R_{t-1,t-1}). \end{aligned}$$

The first term vanishes as  $t \rightarrow \infty$  because  $\theta_t = \alpha\tau_t \rightarrow \alpha\tau_*$  by Proposition 1.2. The second term instead vanishes since  $R_{t,t} \rightarrow \tau_*$ ,  $R_{t,t-1} \rightarrow \tau_*$  by Lemma 5.7.

## D Proof of Lemma 5.2

First note that the upper bound on  $\lambda_{\max}(R/N)$  is trivial since using representations (4.7), (4.8),  $q^t = f_t(h^t, x_0)$ ,  $m^t = g_t(b^t, w)$  and Lemma F.3(c)(d) all entries of the matrix  $R/N$  are bounded as  $N \rightarrow \infty$  and the matrix has fixed dimensions. Hence, we only focus on the lower-bound for  $\lambda_{\min}(R/N)$ .

The result for  $R = M^*M$  and  $R = Q^*Q$  follows directly from Lemma F.3(g) and Lemma 8 of [BM10].

For  $R = Y^*Y$  and  $R = X^*X$  the proof is by induction on  $t$ .



- For  $t = 1$  we have  $Y_t = b^0$  and  $X_t = h^1 + \xi_0 q^0 = h^1 - x_0$ . Using Lemma F.3(b)(c) we obtain almost surely

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{Y_t^* Y_t}{N} &= \delta \lim_{n \rightarrow \infty} \langle b^0, b^0 \rangle = \lim_{N \rightarrow \infty} \langle q^0, q^0 \rangle = \mathbb{E}\{X_0^2\}, \\ \lim_{N \rightarrow \infty} \frac{X_t^* X_t}{N} &= \lim_{N \rightarrow \infty} \langle h^1 - x^0, h^1 - x^0 \rangle = \mathbb{E}\{(\tau_0 Z_0 + X_0)^2\} = \sigma^2 + \frac{\delta + 1}{\delta} \mathbb{E}\{X_0^2\},\end{aligned}$$

where both are positive by the assumption  $\mathbb{P}\{X_0 \neq 0\} > 0$ .

- *Induction hypothesis:* Assume that for all  $t \leq k$  there exist positive constants  $c_X(t)$  and  $c_Y(t)$  such that as  $N \rightarrow \infty$

$$c_Y(t) \leq \lambda_{\min}\left(\frac{Y_t^* Y_t}{N}\right), \quad (\text{D.1})$$

$$c_X(t) \leq \lambda_{\min}\left(\frac{X_t^* X_t}{N}\right). \quad (\text{D.2})$$

- Now we prove Eq. (D.1) for  $t = k + 1$  (proof of (D.2) is similar). We will prove that there is a positive constant  $c$  such that as  $N \rightarrow \infty$ , for any vector  $\vec{a}_t \in \mathbb{R}^t$ :

$$\langle Y_t \vec{a}_t, Y_t \vec{a}_t \rangle \geq c \|\vec{a}_t\|_2^2.$$

First write  $\vec{a}_t = (a_1, \dots, a_t)$  and denote its first  $t - 1$  coordinates with  $\vec{a}_{t-1}$ . Next, we consider the conditional distribution  $A|_{\mathfrak{S}_{t-1}}$ . Using Eqs. (4.9) and (4.10) we obtain (since  $Y_t = A Q_t$ )

$$\begin{aligned}Y_t \vec{a}_t|_{\mathfrak{S}_{t-1}} &\stackrel{d}{=} A|_{\mathfrak{S}_{t-1}}(Q_{t-1} \vec{a}_{t-1} + a_t q^{t-1}) \\ &= E_{t-1}(Q_{t-1} \vec{a}_{t-1} + a_t q^{t-1}) + a_t P_{M_{t-1}}^\perp \tilde{A} q_\perp^{t-1}.\end{aligned}$$

Hence, conditional on  $\mathfrak{S}_{t-1}$  we have, almost surely

$$\lim_{N \rightarrow \infty} \langle Y_t \vec{a}_t, Y_t \vec{a}_t \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \|Y_{t-1} \vec{a}_{t-1} + a_t E_{t-1} q^{t-1}\|^2 + a_t^2 \lim_{N \rightarrow \infty} \langle q_\perp^{t-1}, q_\perp^{t-1} \rangle. \quad (\text{D.3})$$

Here we used the fact that  $\tilde{A}$  is a random matrix with i.i.d.  $\mathcal{N}(0, 1/n)$  entries independent of  $\mathfrak{S}_{t-1}$  (cf. Lemma F.4) which implies that almost surely

$$\begin{aligned}- \lim_{N \rightarrow \infty} \langle P_{M_{t-1}}^\perp \tilde{A} q_\perp^{t-1}, P_{M_{t-1}}^\perp \tilde{A} q_\perp^{t-1} \rangle &= \lim_{N \rightarrow \infty} \langle q_\perp^{t-1}, q_\perp^{t-1} \rangle, \\ - \lim_{N \rightarrow \infty} \langle P_{M_{t-1}}^\perp \tilde{A} q_\perp^{t-1}, Y_{t-1} \vec{a}_{t-1} + a_t b^{t-1} + a_t \lambda_{t-1} m^{t-2} \rangle &= 0.\end{aligned}$$

From Lemma F.3(g) we know that  $\lim_{N \rightarrow \infty} \langle q_\perp^{t-1}, q_\perp^{t-1} \rangle$  is larger than a positive constant  $\varsigma_t$ . Hence, from representation (D.3) and induction hypothesis (D.1)

$$\lim_{N \rightarrow \infty} \langle Y_t \vec{a}_t, Y_t \vec{a}_t \rangle \geq \lim_{N \rightarrow \infty} \left[ \sqrt{c_Y(t-1)} \|\vec{a}_{t-1}\| - \frac{|a_t|}{\sqrt{N}} \|b^{t-1} + \lambda_{t-1} m^{t-2}\| \right]^2 + a_t^2 \varsigma_t.$$

To simplify the notation let  $c'_t \equiv \lim_{N \rightarrow \infty} N^{-1/2} \|b^{t-1} + \lambda_{t-1} m^{t-2}\|$ . Now if  $c'_t |a_t| \leq \sqrt{c_Y(t-1)} \|\vec{a}_{t-1}\|/2$  then

$$\lim_{N \rightarrow \infty} \langle Y_t \vec{a}_t, Y_t \vec{a}_t \rangle \geq \frac{c_Y(t-1)}{4} \|\vec{a}_{t-1}\|^2 + a_t^2 \varsigma_t \geq \min\left(\frac{c_Y(t-1)}{4}, \varsigma_t\right) \|\vec{a}_t\|_2^2, \quad (\text{D.4})$$

which proves the result. Otherwise, we obtain the inequality

$$\lim_{N \rightarrow \infty} \langle Y_t \vec{a}_t, Y_t \vec{a}_t \rangle \geq a_t^2 \varsigma_t \geq \left( \frac{\varsigma_t c_Y (t-1)}{4(c'_t)^2 + c_Y (t-1)} \right) \|\vec{a}_t\|_2^2,$$

that completes the induction argument.

## E A concentration estimate

The following proposition follows from standard concentration-of-measure arguments.

**Proposition E.1.** *Let  $V \subseteq \mathbb{R}^m$  a uniformly random linear space of dimension  $d$ . For  $\lambda \in (0, 1)$ , let  $P_\lambda$  denote the orthogonal projector on the first  $m\lambda$  coordinates of  $\mathbb{R}^m$ . Define  $Z(\lambda) \equiv \sup\{\|P_\lambda v\| : v \in V, \|v\| = 1\}$ . Then, for any  $\varepsilon > 0$  there exists  $c(\varepsilon) > 0$  such that, for all  $m$  large enough (and  $d$  fixed)*

$$\mathbb{P}\{|Z(\kappa) - \sqrt{\lambda}| \geq \varepsilon\} \leq e^{-m c(\varepsilon)}. \quad (\text{E.1})$$

*Proof.* Let  $Q \in \mathbb{R}^{m \times d}$  be a uniformly random orthogonal matrix. Its image is a uniformly random subspace of  $\mathbb{R}^m$  whence the following equivalent characterization of  $Z(\lambda)$  is obtained

$$Z(\lambda) \stackrel{d}{=} \sup\{\|P_\lambda Q u\| : u \in S^d\}$$

where  $S^d \equiv \{x \in \mathbb{R}^d : \|x\| = 1\}$  is the  $d$ -dimensional sphere, and  $\stackrel{d}{=}$  denotes equality in distribution.

Let  $N_d(\varepsilon/2)$  be a  $(\varepsilon/2)$ -net in  $S_d$ , i.e. a subset of vectors  $\{u^1, \dots, u^M\} \in S^d$  such that, for any  $u \in S^d$ , there exists  $i \in \{1, \dots, M\}$  such that  $\|u - u^i\| \leq \varepsilon/2$ . It follows from a standard counting argument [Led01] that there exists an  $(\varepsilon/2)$ -net of size  $|N_d(\varepsilon/2)| \equiv M \leq (100/\varepsilon)^d$ . Define

$$Z_{\varepsilon/2}(\lambda) \equiv \sup\{\|P_\lambda Q u\| : u \in N_d(\varepsilon/2)\}.$$

Since  $u \mapsto P_\lambda Q u$  is Lipschitz with modulus 1, we have

$$\begin{aligned} \mathbb{P}\{|Z(\kappa) - \sqrt{\lambda}| \geq \varepsilon\} &\leq \mathbb{P}\{|Z_{\varepsilon/2}(\kappa) - \sqrt{\lambda}| \geq \varepsilon/2\} \\ &\leq \sum_{i=1}^M \mathbb{P}\{|\|P_\lambda Q u^i\| - \sqrt{\lambda}| \geq \varepsilon/2\}. \end{aligned}$$

But for each  $i$ ,  $Q u^i$  is a uniformly random vector with norm 1 in  $\mathbb{R}^m$ . By concentration of measure in  $S^m$  [Led01], there exists a function  $c(\varepsilon) > 0$  such that, for  $x \in S^m$  uniformly random

$$\mathbb{P}\{|\|P_\lambda x\| - \sqrt{\lambda}| \geq \varepsilon/2\} \leq e^{-m c(\varepsilon)}.$$

Therefore we get

$$\mathbb{P}\{|Z(\kappa) - \sqrt{\lambda}| \geq \varepsilon\} \leq |N_d(\varepsilon/2)| e^{-m c(\varepsilon)} \leq \left(\frac{100}{\varepsilon}\right)^d e^{-m c(\varepsilon)}$$

which is smaller than  $e^{-m c(\varepsilon)/2}$  for all  $m$  large enough.  $\square$

## F Useful reference material

In this appendix we collect a few known results that are used several times in our proof. We also provide some pointers to the literature.

### F.1 Equivalence of $\ell^2$ and $\ell^1$ norm on random vector spaces

In our proof we make use of the following well-known result of Kashin in the theory of diameters of smooth functions [Kas77]. Let  $L_{n,v} = \{x \in \mathbb{R}^n | x_i = 0, \forall i \geq n(1-v) + 1\}$ .

**Theorem F.1** (Kashin 1977). *For any positive number  $v$  there exist a universal constant  $c_v$  such that for any  $n \geq 1$ , with probability at least  $1 - 2^{-n}$ , for a uniformly random subspace  $V_{n,v}$  of dimension  $n(1-v)$ ,*

$$\forall x \in V_{n,v} : \quad c_v \|x\|_2 \leq \frac{1}{\sqrt{n}} \|x\|_1.$$

### F.2 Singular values of random matrices

We will repeatedly make use of limit behavior of extreme singular values of random matrices. A very general result was proved in [BY93] (see also [BS09]).

**Theorem F.2** ([BY93]). *Let  $A \in \mathbb{R}^{n \times N}$  be a matrix with i.i.d. entries such that  $\mathbb{E}\{A_{ij}\} = 0$ ,  $\mathbb{E}\{A_{ij}^2\} = 1/n$ , and  $n = M\delta$ . Let  $\sigma_{\max}(A)$  be the largest singular value of  $A$ , and  $\sigma_{\min}(A)$  be its smallest non-zero singular value. Then*

$$\lim_{N \rightarrow \infty} \sigma_{\max}(A) \stackrel{\text{a.s.}}{=} \frac{1}{\sqrt{\delta}} + 1, \quad (\text{F.1})$$

$$\lim_{N \rightarrow \infty} \sigma_{\min}(A) \stackrel{\text{a.s.}}{=} \frac{1}{\sqrt{\delta}} - 1. \quad (\text{F.2})$$

We will also use the following fact that follows from the standard singular value decomposition

$$\min \{ \|Ax\|_2 : x \in \ker(A)^\perp, \|x\| = 1 \} = \sigma_{\min}(A). \quad (\text{F.3})$$

### F.3 Two Lemmas from [BM10]

Our proof uses the results of [BM10]. We state copy here the crucial technical lemma in that paper. Notations refer to the general algorithm in Eq. (4.1). General state evolution defines quantities  $\{\tau_t^2\}_{t \geq 0}$  and  $\{\sigma_t^2\}_{t \geq 0}$  via

$$\tau_t^2 = \mathbb{E}\{g_t(\sigma_t Z, W)^2\}, \quad \sigma_t^2 = \frac{1}{\delta} \mathbb{E}\{f_t(\tau_{t-1} Z, X_0)^2\}, \quad (\text{F.4})$$

where  $W \sim p_W$  and  $X_0 \sim p_{X_0}$  are independent of  $Z \sim \mathcal{N}(0, 1)$

**Lemma F.3.** *Let  $\{q_0(N)\}_{N \geq 0}$  and  $\{A(N)\}_{N \geq 0}$  be, respectively, a sequence of initial conditions and a sequence of matrices  $A \in \mathbb{R}^{n \times N}$  indexed by  $N$  with i.i.d. entries  $A_{ij} \sim \mathcal{N}(0, 1/n)$ . Assume  $n/N \rightarrow \delta \in (0, \infty)$ . Consider sequences of vectors  $\{x_0(N), w(N)\}_{N \geq 0}$ , whose empirical distributions converge weakly to probability measures  $p_{X_0}$  and  $p_W$  on  $\mathbb{R}$  with bounded  $(2k-2)^{\text{th}}$  moment, and assume:*

- (i)  $\lim_{N \rightarrow \infty} \mathbb{E}_{\hat{p}_{x_0(N)}}(X_0^{2k-2}) = \mathbb{E}_{p_{X_0}}(X_0^{2k-2}) < \infty.$
- (ii)  $\lim_{N \rightarrow \infty} \mathbb{E}_{\hat{p}_{w(N)}}(W^{2k-2}) = \mathbb{E}_{p_W}(W^{2k-2}) < \infty.$
- (iii)  $\lim_{N \rightarrow \infty} \mathbb{E}_{\hat{p}_{q_0(N)}}(X^{2k-2}) < \infty.$

Let  $\{\sigma_t, \tau_t\}_{t \geq 0}$  be defined uniquely by the recursion (F.4) with initialization  $\sigma_0^2 = \delta^{-1} \lim_{n \rightarrow \infty} \langle q^0, q^0 \rangle$ . Then the following hold for all  $t \in \mathbb{N} \cup \{0\}$

(a)

$$h^{t+1}|_{\mathfrak{S}_{t+1,t}} \stackrel{d}{=} \sum_{i=0}^{t-1} \alpha_i h^{i+1} + \tilde{A}^* m_{\perp}^t + \tilde{Q}_{t+1} \vec{o}_{t+1}(1), \quad (\text{F.5})$$

$$b^t|_{\mathfrak{S}_{t,t}} \stackrel{d}{=} \sum_{i=0}^{t-1} \beta_i b^i + \tilde{A} q_{\perp}^t + \tilde{M}_t \vec{o}_t(1), \quad (\text{F.6})$$

where  $\tilde{A}$  is an independent copy of  $A$  and the matrix  $\tilde{Q}_t$  ( $\tilde{M}_t$ ) is such that its columns form an orthogonal basis for the column space of  $Q_t$  ( $M_t$ ) and  $\tilde{Q}_t^* \tilde{Q}_t = N \mathbf{I}_{t \times t}$  ( $\tilde{M}_t^* \tilde{M}_t = n \mathbf{I}_{t \times t}$ ).

(b) For all pseudo-Lipschitz functions  $\phi_h, \phi_b : \mathbb{R}^{t+2} \rightarrow \mathbb{R}$  of order  $k$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi_h(h_i^1, \dots, h_i^{t+1}, x_{0,i}) \stackrel{\text{a.s.}}{=} \mathbb{E}\{\phi_h(\tau_0 Z_0, \dots, \tau_t Z_t, X_0)\}, \quad (\text{F.7})$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_b(b_i^0, \dots, b_i^t, w_i) \stackrel{\text{a.s.}}{=} \mathbb{E}\{\phi_b(\sigma_0 \hat{Z}_0, \dots, \sigma_t \hat{Z}_t, W)\}, \quad (\text{F.8})$$

where  $(Z_0, \dots, Z_t)$  and  $(\hat{Z}_0, \dots, \hat{Z}_t)$  are two zero-mean gaussian vectors independent of  $X_0, W$ , with  $Z_i, \hat{Z}_i \sim \mathbf{N}(0, 1)$ .

(c) For all  $0 \leq r, s \leq t$  the following equations hold and all limits exist, are bounded and have degenerate distribution (i.e. they are constant random variables):

$$\lim_{N \rightarrow \infty} \langle h^{r+1}, h^{s+1} \rangle \stackrel{\text{a.s.}}{=} \lim_{n \rightarrow \infty} \langle m^r, m^s \rangle, \quad (\text{F.9})$$

$$\lim_{n \rightarrow \infty} \langle b^r, b^s \rangle \stackrel{\text{a.s.}}{=} \frac{1}{\delta} \lim_{N \rightarrow \infty} \langle q^r, q^s \rangle. \quad (\text{F.10})$$

(d) For all  $0 \leq r, s \leq t$ , and for any Lipschitz function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ , the following equations hold and all limits exist, are bounded and have degenerate distribution (i.e. they are constant random variables):

$$\lim_{N \rightarrow \infty} \langle h^{r+1}, \varphi(h^{s+1}, x_0) \rangle \stackrel{\text{a.s.}}{=} \lim_{N \rightarrow \infty} \langle h^{r+1}, h^{s+1} \rangle \langle \varphi'(h^{s+1}, x_0) \rangle, \quad (\text{F.11})$$

$$\lim_{n \rightarrow \infty} \langle b^r, \varphi(b^s, w) \rangle \stackrel{\text{a.s.}}{=} \lim_{n \rightarrow \infty} \langle b^r, b^s \rangle \langle \varphi'(b^s, w) \rangle. \quad (\text{F.12})$$

Here  $\varphi'$  denotes derivative with respect to the first coordinate of  $\varphi$ .

(e) For  $\ell = k - 1$ , the following hold almost surely

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (h_i^{t+1})^{2\ell} < \infty, \quad (\text{F.13})$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (b_i^t)^{2\ell} < \infty. \quad (\text{F.14})$$

(f) For all  $0 \leq r \leq t$ :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle h^{r+1}, q^0 \rangle \stackrel{\text{a.s.}}{=} 0. \quad (\text{F.15})$$

(g) For all  $0 \leq r \leq t$  and  $0 \leq s \leq t - 1$  the following limits exist, and there exist strictly positive constants  $\rho_r$  and  $\varsigma_s$  (independent of  $N, n$ ) such that almost surely

$$\lim_{N \rightarrow \infty} \langle q_{\perp}^r, q_{\perp}^r \rangle > \rho_r, \quad (\text{F.16})$$

$$\lim_{n \rightarrow \infty} \langle m_{\perp}^s, m_{\perp}^s \rangle > \varsigma_s. \quad (\text{F.17})$$

It is also useful to recall some simple properties of gaussian random matrices.

**Lemma F.4.** For any deterministic  $u \in \mathbb{R}^N$  and  $v \in \mathbb{R}^n$  with  $\|u\| = \|v\| = 1$  and a gaussian matrix  $\tilde{A}$  distributed as  $A$  we have

(a)  $v^* \tilde{A} u \stackrel{d}{=} Z / \sqrt{n}$  where  $Z \sim \mathcal{N}(0, 1)$ .

(b)  $\lim_{n \rightarrow \infty} \|\tilde{A} u\|^2 = 1$  almost surely.

(c) Consider, for  $d \leq n$ , a  $d$ -dimensional subspace  $W$  of  $\mathbb{R}^n$ , an orthogonal basis  $w_1, \dots, w_d$  of  $W$  with  $\|w_i\|^2 = n$  for  $i = 1, \dots, d$ , and the orthogonal projection  $P_W$  onto  $W$ . Then for  $D = [w_1 | \dots | w_d]$ , we have  $P_W A u \stackrel{d}{=} D x$  with  $x \in \mathbb{R}^d$  that satisfies:  $\lim_{n \rightarrow \infty} \|x\| \stackrel{\text{a.s.}}{=} 0$  (the limit being taken with  $d$  fixed). Note that  $x$  is  $\vec{o}_d(1)$  as well.

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